Of all the types of waves we study, we are most familiar with water waves as seen in oceans, lakes, rivers, and bathtubs. We’re also familiar with waves created by air currents through fields of grasses or wheat. In reality, we constantly experience waves of various types. Sound, light, radio, and other forms of electromagnetic radiation surround us every moment of our lives and although we do not directly “see” their waves, aside from visible light, these phenomena can all be understood in terms of waves. Furthermore, we show later that matter also behaves as a wave and that our current quantum physics picture of the world is intimately connected with a mathematical description known as the wave function. Waves are thus the key to our understanding of nature on a fundamental level.

In this chapter we first return to the type of motion known as simple harmonic motion that we used to describe a mass on a spring in Chapter 3. Here we extend our previous discussions to include the frictional loss of energy, known as damping, and the effects of a “driving force” used to sustain the motion. With the addition of energy by this external force comes the possibility of a resonance phenomenon in which the amplitude of oscillation can grow rapidly. This is an extremely important idea in physics that we will see often throughout the rest of our studies. We then introduce some fundamental concepts concerning waves and consider traveling waves along a string and along a coiled spring as mechanical examples of the two basic forms of waves, transverse and longitudinal. As waves travel along or through a medium, they meet and interact with boundaries or obstacles, and different interactions possible at a boundary are considered, including reflection and refraction. We also discuss one possible result from such boundary conditions, the creation of standing waves. These are important in such diverse areas as musical instruments, the human ear, and the basic functioning of a laser, all considered later in this book.

1. SIMPLE HARMONIC MOTION REVISITED: DAMPING AND RESONANCE

A linear restoring force is the basis of simple harmonic motion. Our example has been the spring force, \( F = -kx \), first studied in Chapter 3. The characteristic of simple harmonic motion is the variation in oscillator position according to

\[
x(t) = A \cos(\omega_0 t),
\]

where \( \omega_0 \) is the angular frequency that depends on the parameters of the particular type of simple harmonic oscillator. For example, in the case of a mass on a spring we have seen that the angular frequency is given by \( \omega_0 = \sqrt{\frac{k}{m}} \). We have already introduced the definitions of the frequency, \( f \), and period, \( T \), which are related to the angular frequency in general by

\[
f_0 = \frac{1}{T} = \frac{\omega_0}{2\pi}
\]
The frequency \( f_0 \) is often called the natural frequency of oscillation of the isolated system because it is the frequency the system adopts if released and left unperturbed. Later in this section we consider cases when an external force, oscillating at a frequency different from the natural frequency, acts on the system (Figure 10.1). As we have also seen when we considered potential energy, the energy of a simple harmonic oscillator remains constant, exchanging periodically between kinetic and potential energy.

A second example of an oscillating system that can be modeled as undergoing simple harmonic motion is the so-called simple pendulum, consisting of a point mass suspended from a massless string or rod of length \( L \). A true simple pendulum consists of a mass with dimensions small compared to \( L \) and a light string or rod. If the pendulum is made to oscillate in a plane, we can show that if the string makes a small (\(<10^\circ\)) angle with the vertical that this angle will oscillate according to Equation (10.1) with \( x \) replaced by the angle \( \theta \) equal to the maximum angle, and \( \omega_0 \) given by

\[
\omega_0 = \sqrt{\frac{L}{g}}.
\]

Thus, the motion of the simple pendulum is independent of its mass, depending only on its length.

Simple harmonic motion is an abstraction. All real oscillators lose energy over time due to frictional forces. This was first seen in the Chapter 3 section on viscoelasticity where we discussed models in which the elastic springs were combined with frictional dashpots to describe the viscous effects of the material. Let’s now consider in more detail the effect of frictional forces on the simple harmonic motion of a mass on a spring. We model the frictional (damping) force as linearly dependent on the velocity of the mass. This is a good approximation when the damping forces are small. Then the net force on the mass is given by

\[
F_{\text{net}} = -kx - bv,
\]

where \( b \) is a frictional or damping constant.

What is the effect of this damping on the motion of the mass? If the damping is small we might guess correctly that the resulting motion would be an oscillation with slowly decreasing amplitude. The correct expression for the oscillator position with damping is

\[
x(t) = (Ae^{-\frac{bt}{2m}}) \cos(\omega_{\text{damp}}t),
\]

where

\[
\omega_{\text{damp}} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.
\]
where the angular frequency is a constant somewhat different than in the case of no damping and given by

$$\omega_{\text{damp}} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \quad (10.5)$$

Note that if $b = 0$ this expression reduces to the angular frequency in the absence of damping, as it must. The first term in parentheses in Equation (10.4) is an exponentially decreasing amplitude. Figure 10.2 shows a typical graph of Equation (10.4); the dashed lines are called the envelope of the equation and show the exponentially decreasing amplitude of oscillation.

The energy of a spring undergoing undamped simple harmonic motion is equal to the constant value $\frac{1}{2}kA^2$. The energy of the damped oscillator can be found by substituting the exponentially decreasing amplitude to find

$$E = \frac{1}{2}kA^2e^{-\frac{bt}{m}}, \quad (10.6)$$

that itself decreases exponentially with time. Thus, once made to oscillate, a damped harmonic oscillator will maintain a fixed period of oscillation, given by $T = \frac{2\pi}{\omega_{\text{damp}}}$, but will have an amplitude and energy that continuously decrease (Figure 10.3). The damped harmonic oscillator model can be used to describe many other systems in addition to springs. For example, molecules that interact with each other but lose...
energy via collisions or other mechanisms can also be modeled using spring and damping constants that can be related to the interaction parameters. Also a real pendulum with damping forces can be modeled in a parallel way.

Example 10.1 A 0.2 kg mass is attached to a spring with a spring constant of \( k = 40 \text{ N/m} \) and a damping constant of \( b = 0.02 \text{ kg/s} \) and allowed to come to equilibrium. If the spring is then stretched a distance of 10 cm and released from rest, find the following: (a) the initial energy; (b) the natural frequency; (c) the actual period of the motion; (d) the time for the amplitude to decrease to 5 cm, half of its initial value; and (e) the time for half the energy to be dissipated.

Solution: (a) The initial energy is equal to \( \frac{1}{2} k A^2 \) (this is also the \( t = 0 \) value of energy obtained from Equation (10.6)) and is therefore \( E_i = 0.5(40)(.1)^2 = 0.2 \text{ J} \).

(b) The natural frequency is defined by Equation (10.2). Recalling that for a spring, we have that

\[
\omega_0 = \sqrt{\frac{k}{m}}
\]

so that

\[
f_0 = \sqrt{\frac{40}{2}} / 2\pi = 2.25 \text{ Hz}.
\]

(c) The actual period of the motion is \( T = 2\pi/\omega \), where \( \omega \) is the actual angular frequency of the oscillation, affected by the damping, and given by Equation (10.5).

We have that \( T = 2\pi\sqrt{\frac{40}{2} - \frac{0.02^2}{4 \cdot 0.2^2}} = 0.44 \text{ s} \). This value is extremely close to the period in the absence of damping; the second term in the square root is negligible; in fact, in order for that term, \( b^2/4m^2 \), to make a 5% change in the period, \( b \) must as large as 0.2 kg/s.

(d) Because the amplitude decays exponentially, we can write from Equation (10.4) that \( A(t) = A(0)e^{-bt/2m} \). Substituting we have \( 0.05 = 0.1 e^{-0.02t/(2)(0.2)} = 0.1 e^{-0.05t} \), or \( 0.5 = e^{-0.05t} \). We solve this equation by taking the natural logarithm of both sides of the equation: \( \log 0.5 = \log(e^{-0.05t}) = -0.05 t \), so that \( t = -(\log 0.5)/0.05 = 13.9 \text{ s} \).

(e) The time for half the energy to be dissipated is found in a similar way using Equation (10.6) in the form \( E(t) = E(0) e^{-bt/m} \). Because we want the time for \( E(t)/E(0) = 0.5 \), we write \( 0.5 = e^{-bt/m} \) and again take the natural logarithm of both sides, to find \( t = -(\log(0.5))m/b = 6.9 \text{ s} \), or half the time for the amplitude to drop to half its starting value, as expected from the factor of two difference in the exponents.

In practice, oscillators have their amplitude maintained by adding energy from the outside; for example, the pendulum on a grandfather clock maintains its amplitude of oscillation from the energy of a spring or a mechanical gear mechanism that requires winding. We can account for an external force \( F_{\text{ext}} \) by adding a term to Equation (10.3) so that the net force on the oscillator mass is now

\[
F_{\text{net}} = -kx - bv + F_{\text{ext}}
\]

If the external force is sinusoidal, with a frequency \( f_{\text{ext}} \) known as the external driving frequency then, after sufficient time to reach a steady state in which the motion remains periodic, the oscillator position is given as

\[
x(t) = A(\omega_0 \cdot \omega_{\text{ext}}) \cos(\omega_{\text{ext}} t + \varphi),
\]
where the amplitude $A$ depends on both the natural angular frequency of the oscillator $\omega_0$ and that of the external driving force, the oscillation frequency is that of the external force, but a phase shift $\varphi$ appears so that the driving force and oscillator response are not necessarily in synchrony in time.

After reaching this steady-state condition, the energy added to the oscillator by the driving force in one cycle of oscillation must equal the energy loss through dissipation by the frictional damping force $(-b\ddot{v})$ in that same period of time $T$. The input energy in one cycle is given by the product of the power and the period $E = (\dot{F}_{\text{ext}}, \dot{v})T$, where the input power is averaged over one cycle of time. If this input energy is small, then the dissipation (velocity term) must be equally small, and so the velocity and hence the oscillator amplitude will be correspondingly small. On the other hand, if the energy input is large then the dissipation must be large, so that the oscillator velocity and therefore amplitude will also be large. We call this phenomenon resonance.

What controls the average energy input? Well, clearly the strength of the driving force will be a factor here. However, for a given driving force amplitude, what determines the energy input is how close its driving frequency is to the natural frequency of the oscillator. This is true because the energy input depends on $\dot{F}$ and $\dot{v}$ pointing in the same direction and since both are sinusoidal functions oscillating in the $+x$ and $-x$ directions, as is shown in Figure 10.4, if their two frequencies are very different, the average time they are pointed in the same direction will be much smaller than if their two frequencies are close. Quantitatively, the amplitude of the driven damped harmonic oscillator is given by

$$A = \frac{F/m}{\sqrt{(\omega_0^2 - \omega_{\text{ext}}^2)^2 + \left(b\omega_{\text{ext}}/m\right)^2}}. \quad (10.9)$$

Figure 10.5 shows how the amplitude depends on the external driving force. A pronounced maximum, or resonance, occurs as the driving frequency approaches the natural frequency of the oscillator.

Every day examples of resonance abound. When a child on a swing is pushed by a friend, maximum amplitude is reached when the pushes come in sync.
with the natural frequency of oscillation. Hikers marching in step over a suspended bridge can cause large amplitude vibrations of the bridge. Occasionally a similar phenomenon will destroy a poorly designed bridge or highway when energy from wind or earthquakes causes large amplitude oscillations that can weaken the structure. This was the cause of a major highway collapse during the 1989 earthquake in the San Francisco Bay area, for example (Figure 10.6). We also show in the next chapter that resonance plays a major role in the design of musical instruments as well as in the sensitivity of our ears to different frequencies of sound. Many electronic circuits have resonances; when you tune a radio or change the channel on a TV you are choosing a particular resonant frequency. A variety of biophysical techniques also involve resonances, including nuclear magnetic resonance (NMR, and its imaging version, magnetic resonance imaging or MRI), and electron spin resonance (ESR).

2. WAVE CONCEPTS

Mechanical waves are vibrational disturbances that travel through a material medium (in this section we assume no energy dissipation). Examples include water waves, sound waves traveling in a medium such as air or water, waves along a string (as in a musical instrument) or along a steel beam, or seismic waves traveling through the Earth. A general characteristic of all waves is that they travel through a material medium (except for electromagnetic waves which can travel through a vacuum) at characteristic speeds over extended distances; in contrast, the actual molecules of the material medium vibrate about equilibrium positions at different characteristic speeds, and do not translate along the wave direction.

Mechanical waves on a stretched string can be directly visualized. Imagine that we tie one end of a string to a fixed point and stretch it tightly. We can send a wave pulse down the string by giving the held end a single rapid up and down oscillation (Figure 10.7). The motion of the string is vertical whereas the pulse travels horizontally along the string. The vertical forces acting from one region of the string to the next near the leading edge of the pulse are what sustain the pulse and cause it to move along the string. If we continue to oscillate the held end at a fixed frequency \( f \), then we set up a series of identical oscillations, or a periodic wave, that travels down the string (Figure 10.8). Such waves are called transverse, because the medium oscillates in a plane perpendicular to the direction in which the wave travels.

Suppose we replace the string by a stretched spring tied at one end. If we oscillate the free end of the spring either once, or continuously, along the horizontal direction (along its axis), we set up a longitudinal pulse, or periodic wave, in which the motion of the material medium is an oscillation along the direction of propagation of the wave (Figure 10.8).

From a flash photo at some instant of time of the string undergoing continuous oscillations, we can see that the wave consists of a repeating series of positive (above axis, where the axis is the unperturbed string) and negative (below axis) pulses. The distance between corresponding points of one pulse and the next is called the wavelength, \( \lambda \). Because the waveform, or shape, is repetitive, or periodic, corresponding points can be neighboring maxima, crests, of the wave, or minima, troughs, of the wave, or any set of neighboring corresponding points (Figure 10.9).
A similar analysis applies to the longitudinal waves of the spring, where now positive and negative refer to the compression or extension of the spring compared to its unperturbed configuration. In this case it is easier to see the wave variation with time clearly by performing the intermediate step of graphing the longitudinal displacement as a function of time to obtain a curve similar to Figure 10.9.

As a wave moves along the string, we can ask with what speed it is traveling. If we look at an arbitrary point along the string, we will see exactly one wave move by in a period, the time $T = 1/f$ required for one oscillation. The distance the wave travels in this time is exactly one wavelength. Therefore, the velocity of the wave is given, quite generally, by

$$v = \frac{\lambda}{T} = \lambda f.$$  \hfill (10.10)

This same expression holds for longitudinal waves as well and is applicable to all types of waves, from mechanical to electromagnetic.

In addition to mechanical waves on a string or spring, there are several important examples of other waves that we study in this book. Sound waves are mechanical pressure waves traveling in an elastic medium, fluid or solid, causing density variations with regions of lower and higher density. In a solid these waves can be both transverse and longitudinal (as in an earthquake when seismic waves travel through the Earth), but in a fluid, such as air or water, sound waves are only longitudinal. Water waves are also a combination of transverse and longitudinal waves that produce a rolling motion so that as a wave passes by, the water actually travels in an elliptical path. (If you’ve ever floated in the ocean surf, you will remember that your motion is both up and down as well as horizontal so that you periodically oscillate in a looplike rolling motion.) Electromagnetic waves are transverse waves that are studied in some detail later where we show that these waves do not require a medium in which to propagate but can travel through a vacuum at the speed of light.

Every type of periodic wave has its source in some periodic vibration. For example, sound may be produced by the vibrations of a string, a membrane (drumhead), an air column, or a tuning fork; vibrations of electrons can produce electromagnetic waves of a variety of types including visible light and radio waves. Furthermore, different types of waves will interact with matter in different ways that we study in the course of the remainder of this book.

Waves that can be described by a sinusoidal variation are called harmonic waves. At any fixed position such waves vary with time according to Equation (10.1). The wave will also vary with position at a fixed time. For waves on a string, the spatial variation at a fixed time can be captured by a snapshot of a harmonic wave frozen in time that would appear as a sinusoidal curve. We could then describe the vertical position of the string measured from its equilibrium horizontal position in the snapshot photo as

$$y(x) = A \sin(kx),$$  \hfill (10.11)

where $k$, known as the wave number, is related to the wavelength through the relation

$$k = \frac{2\pi}{\lambda}. \hfill (10.12)$$
Thus as we move horizontally along the snapshot of the string, in the x-direction, the vertical variation in the height of the string is sinusoidal with an amplitude A and a spatial repeat distance of λ (Figure 10.10). Because the sine function has a period of $2\pi$ radians, writing the argument as $2\pi x/\lambda$ ensures that each time x increases its value by λ, the argument of the sine function will have increased by $2\pi$, maintaining the same value for the function $y(x)$.

Each point on the string actually oscillates in the vertical direction as time goes by so that y, the vertical coordinate, varies not only with x, the distance along the string, but also with time. This is a generalization of Equation (10.1) in which a one-dimensional harmonic oscillator was described using $x(t)$. For a wave on a string the y-coordinate of each point along the string (with a different x-coordinate) varies in time according to an equation similar to Equation (10.1) but with $x$ replaced by $y$. In the next section we show how we can connect the motion of each point along the string in a simple mathematical way.

3. TRAVELING WAVES

The frozen-in-time snapshot of a sinusoidal wave on a string in the last section actually is traveling along the string in a way that maintains the shape of the wave as it moves along the string. We can describe such a traveling harmonic wave mathematically by writing an expression for the vertical displacement of the string as a function of both $x$, the horizontal position along the string, and $t$, the time, as

$$y(x,t) = A \sin (kx - \omega t), \quad (10.13)$$

where $\omega$ is the angular frequency of oscillation (remember that $\omega = 2\pi f$). In this section we ignore what happens to the wave at the end of the string by imagining the string to be very long. We consider the effects of a boundary, for example, the tied end of the string, in the next section.

Let's consider the meaning of Equation (10.13) more carefully. If we fix the value of $t$, then we are looking at the spatial variation of the wave frozen in time as we just did in the last section. Different constant nonzero values of $t$ in Equation (10.13) simply shift the argument of the sine function in Equation (10.11) without any other changes. Note that for a wave to travel along a string, the string must be elastic, or able to stretch. That this is so is obvious on considering that the contour length along the sine curve is clearly greater than the straight line distance along the string axis. The stretch of the string varies along its length and is proportional to the slope of the string. Where the slope is greatest, at the $y = 0$ crossings or nodes, the string is stretched the most, however, where the slope is zero, at the amplitude where $y$ is a maximum or minimum, the string is unstretched.

If we fix, instead of time $t$, the value of $x$ so that we are looking at the time dependence of the wave at a fixed point on the string, Equation (10.13) reveals a sinusoidal oscillation of the string up and down with an amplitude $A$ and an angular frequency $\omega$ or period $T$ (Figure 10.11). Each element of the string moves only vertically. This is precisely the motion of the string to be expected as the waveform given by Equation (10.11) moves by with a velocity $v$. In this case the waveform remains constant but moves along the positive x-direction at a velocity such as to keep the argument $(kx - \omega t)$, and hence $y$, equal to a constant. This will occur if $v = x/t = \omega/k = (2\pi f)/(2\pi\lambda) = \lambda f$, in agreement with Equation (10.10). Thus as the clock ticks on and $t$ increases, the entire waveform, representing $y(x, t)$ moves along the positive x-axis at velocity $v$. In the case of a wave traveling toward the negative x-axis, the argument in Equation (10.13) simply gets replaced by $(kx + \omega t)$, so that there is a negative velocity with the same magnitude as that in Equation (10.10).

What determines the frequency and wavelength of the waves traveling along the string? In the case we have been discussing in which one end of the string is made to oscillate, the frequency is determined by the external driving frequency. The wave
velocity for small amplitude waves is determined by two quantities: the tension in the string $F_T$ and an intrinsic property of the string, its mass density or mass per unit length, according to

$$v_{\text{wave}} = \sqrt{\frac{F_T}{(m/L)}}. \quad (10.14)$$

The wavelength of the traveling waves is then determined by the frequency and the wave speed, according to Equation (10.10). From this discussion, we expect that the greater the tension is in the string, the faster the waves travel, and, for a given frequency of oscillation, the longer the wavelength. Similarly, for the same driving frequency and length of string, a more massive string will result in a slower wave speed and therefore a shorter wavelength.

**Example 10.2** A traveling wave on a string is described by the equation $y = 0.025 \sin(1.5x - 200t)$ where $x$ and $y$ are measured in m and $t$ in s. If the string has a mass per unit length of 0.003 kg/m, find the following quantities: the amplitude, wavelength, frequency, period, the velocity of the wave, and the tension in the string.

**Solution:** From the general form of a traveling wave on a string, given by Equation (10.13), we can identify directly from the given equation that the amplitude $A = 0.025$ m, the wave number $k = 2\pi/\lambda = 1.5$ m$^{-1}$, and the angular frequency $\omega = 2\pi/T = 200$ rad/s. We can therefore straightforwardly compute the wavelength to be $\lambda = 2\pi/k = 4.2$ m and the period to be $T = 2\pi/\omega = 0.031$ s. The frequency is the inverse of the period and is therefore equal to $f = 1/T = 31.8$ Hz. Because the wave travels a distance of one wavelength in a time equal to one period, the wave velocity is given as $v = \lambda/T = 130$ m/s. From this value and the equation connecting the speed of a wave on a string to the tension in the string (Equation (10.14)), we can solve for the tension, $F_T = v^2(m/L) = 53$ N.

Having described the waveform, the relationships between the variables describing the waveform and the velocity of a wave on a string, we can ask the obvious question: if a wave is not the translational motion of the material medium itself, what is it that is transported with the wave velocity? The answer is energy. Continuing with our example of the string, the energy that is input to the system from the external driving force at one end is transmitted along the string at velocity $v_{\text{wave}}$. With a single pulse sent down the string it is clear that the kinetic energy of the transverse motion of the string is translated along the string with the pulse. If we imagine the string to be divided up into short segments along the $x$-direction, we can ask where the segments have their maximum and minimum kinetic and potential energy when a harmonic wave travels along the string. Because each element moves vertically, oscillating harmonically about $y = 0$ as a function of time, the kinetic energy of an element is a maximum as it moves through the $y = 0$ position (Figure 10.12).

At the amplitude, $y = \pm A$, the segment is instantaneously at rest and therefore has no kinetic energy. The stretch of the string is proportional to its slope, and the elastic potential energy is proportional to the product of the tension force and the stretch, therefore we see that the elastic potential energy is also maximum at $y = 0$ where the slope of the string is a maximum. Again at the amplitude, $y = \pm A$, the slope of the string is zero and therefore so is the elastic potential energy. (Note that this is in contrast to a mass on a spring, where the elastic potential energy is a maximum at the amplitude.) In fact, it can be shown that the kinetic and potential energies are exactly equal for harmonic waves traveling along an elastic string, with the peaks in energy located at the $y = 0$ crossings and moving with the wave velocity.
along the string. So it is energy that travels along the string and constitutes the wave. The elements of the string behave as harmonic oscillators each carrying a total energy proportional to the square of the wave amplitude and transmitting that energy along the string through the elastic interactions with neighboring string elements. This is a general result of harmonic waves: the total energy carried by the waves is proportional to the square of the wave amplitude.

We have only discussed traveling waves along an elastic string. Traveling longitudinal harmonic waves can also be produced on a coiled spring by oscillating one end longitudinally at a fixed frequency (see Figure 10.8). The variations in the compression and expansion of the spring result in a wave traveling down the spring. If \( y(x, t) \) represents the local displacement (assumed small) of the spring from its equilibrium position as a function of both the position along the spring, \( x \), and the time, \( t \), then Equation (10.13) fully describes such longitudinal waves as well.

Both of the examples of traveling waves cited are one-dimensional cases with waves traveling along the \( x \)-direction. When a rock is dropped in a pond of water, waves spread out radially along the two-dimensional surface of the pond with the wavefronts (or shape of the crests) forming circles. Light waves from a light bulb travel radially outward in space in three dimensions with spherical wavefronts, as do sound waves from a person who is speaking. We study some of these examples later in the text, but we note that the fundamental definitions introduced in this chapter are still appropriate but that our one-dimensional pictures need to be generalized for these other situations.

### 4. WAVES AT A BOUNDARY: INTERFERENCE

When traveling waves reach boundaries between two different media several different phenomena can occur. In the case of one-dimensional waves, at a boundary part of the incident wave will continue into the new medium as the transmitted wave, traveling at a different velocity due to the medium’s different properties, and the balance of the wave’s energy will be reflected back within the incident medium as the reflected wave. In the case of waves traveling in the positive \( x \)-direction along a string with a particular linear mass density \( m/L \) tied to another string with a different mass density at a knot between the two strings, the knot serves as the boundary. As the incident wave (or pulse) arrives at the boundary, there will be both a transmitted and a reflected wave (pulse). A portion of the energy will enter the new medium and the transmitted wave (pulse) will continue to travel in the positive \( x \)-direction but at a different velocity according to Equation (10.14). \( (F_T \) will be the same but \( m/L \) is different.) The reflected wave (pulse) will contain the balance of the incident energy and will return along the string traveling along the negative \( x \)-direction.
Consider the example of an individual pulse traveling to the right along our string as shown in Figure 10.13. The figure shows the time sequence of events that occur when this pulse reaches the knot between two different strings. A portion of the amplitude of the pulse continues into the second string traveling to the right. The reflected pulse passes through the incident pulse emerging in reverse order traveling to the left. During the time that the two pulses overlap along the string their amplitudes are seen to add together. This is an example of the superposition principle, an extremely important concept in wave physics. We have already seen the superposition principle in action when, in Chapter 5, we noted that the net vector force was the sum of the individual vector forces acting on an object. For waves, this principle states that the wave displacement at any point is the algebraic sum of the individual displacements of the overlapping waves at that point. Said differently, the net waveform is the algebraic sum of the individual waveforms.

A consequence of the superposition principle is the phenomenon known as interference. Two transverse waves traveling in the same direction along the same string will add together to produce a resultant wave that is the observed waveform. Mathematically the expressions for the two waves add algebraically. If they have the same wavelength (and, because the velocities are the same, also the same frequency) and are in phase, so that their crests and troughs march together along the string, then the resultant amplitude will be their sum. In this case if the two waves are identical, each of amplitude $A$, the resultant wave will have an amplitude of $2A$ (Figure 10.14 left). These two waves are said to combine by constructive interference.

If the waves have equal amplitude $A$, and the resulting waves are completely out of phase, so that the crest of one travels together with the trough of the other, then the two waves combine by destructive interference and, in this case of equal amplitudes, will completely eradicate each other resulting in no disturbance of the string at all. If the two out of phase waves have different amplitudes $A_1$ and $A_2$, as in the center panel of Figure 10.14, then the destructive interference leads to partial cancellation of the waves and an amplitude equal to $|A_1 - A_2|$. When the two waves are partially out of phase, as in the right panel of Figure 10.14, they will add together to produce a wave with the same wavelength but an amplitude that is between 0 and $A_1 + A_2$ (if the amplitudes are equal) depending upon their phase difference (or relative position of their crests).

We can explore the interference of two equal amplitude waves a bit further by writing each of the two waves that overlap in the form of Equation (10.13), but with one wave shifted by an arbitrary phase $\varphi$ with respect to the other so that

$$y_1 = A \sin (kx - \omega t) \quad \text{and} \quad y_2 = A \sin (kx - \omega t + \varphi). \quad (10.15)$$
When these two waves overlap, the principle of superposition tells us that the total wave amplitude will be

\[ y = y_1 + y_2 = A\sin(kx - \omega t) + \sin(kx - \omega t + \varphi). \]

Using a trigonometric identity (namely, \( \sin \alpha + \sin \beta = 2\sin \frac{\alpha + \beta}{2}\cos \frac{\alpha - \beta}{2} \)), we can simplify this expression to find

\[ y = [2\cos \frac{1}{2} \varphi] \sin (kx - \omega t + \frac{1}{2} \varphi). \]  \hspace{1cm} (10.16)

This result shows that the superposition is also a traveling wave with the same wavelength and frequency, but shifted in phase by \( \varphi/2 \) and with an amplitude, given by the terms in the square bracket, that depends on the phase angle and lies between 0 and 2A.

If the two traveling waves are in phase, or interfere constructively with \( \varphi = 0 \), then Equation (10.16) yields a net amplitude equal to the sum of the separate amplitudes (2A), as we saw earlier. On the other extreme, if the two traveling waves interfere completely destructively with \( \varphi = \pi \) or 180°, then the two waves will exactly cancel, giving an amplitude identically equal to 0. Equation (10.16) gives the result for the general case of arbitrary phase angle.

As a further example of interference, consider the case of two waves of slightly different wavelength (or frequency) traveling in the same direction along the same string. Figure 10.15 shows two waves that differ in frequency by 10% (red and
green sine curves) and their superposition (in blue). Notice that in addition to a periodic variation at the average frequency, the resultant wave has a slower periodic variation that occurs at the difference frequency. In the figure this lower frequency component has a period equal to ten times that of the higher frequency component; you can count ten peaks between a longer period repeat. The slower variation is due to the interference of the two waves that leads to more or less cancellation in a period fashion. This phenomenon is known as beats and in the case of sound waves results in an audible low frequency variation in loudness. When two tones are played that are very close in frequency, one hears the average frequency tone modulated in loudness at the difference or beat frequency. This phenomenon is discussed in more detail in Section 3 of the next chapter. Beats can be used to tune an instrument when a standard frequency is used to generate one of the tones; the instrument is tuned so as to lower the beat frequency, lengthening the period of the loudness variations. In the limit of an infinite beat period the two frequencies are identical.

5. STANDING WAVES AND RESONANCE

We now consider the situation on a string when we force one end to oscillate in simple harmonic motion at some frequency $f$ and fix the other end of the string so that it cannot move. In this case as the wave reaches the fixed end, all of its energy is reflected, and the reflected wave reverses its sign. This reversal of sign is a byproduct of the requirement of a fixed point; if the wave did not reverse itself on reflection then the amplitude would not always add to zero at the fixed point. Reversal of sign of the reflected wave also occurs for the case of two strings tied together when the wave travels from the lighter to the heavier string, a situation shown in Figure 10.13.

If the string has a length $L$, then the round-trip of the wave down and back along the string requires a time equal to $2L/v_{\text{wave}}$. If the reflected wave arrives back at the oscillating end of the string at a time just equal to a period of oscillation $1/f$ of the driven end of the string, then the waves traveling to the right and the left will be exactly in phase and constructively interfere, producing a standing wave as shown in the sequence of events in Figure 10.16.

We can understand this result by adding together two waves of equal amplitude that are traveling along the string in opposite directions. Given

$$y_1 = A \sin(kx + \omega t) \quad \text{and} \quad y_2 = A \sin(kx - \omega t),$$

![Figure 10.16](image)

**FIGURE 10.16** A sequence of eight equal time views spanning one period and showing a string tied down at the right and driven at the left. A wave pulse travels to the right (blue) in the first four views, reaching the knot. In subsequent views, a reflected wave (red dashed curve or red arrow) returns to the left, and the incident wave continues to the right; the black curves are the superposition of the incident (blue) and reflected (red) waves and may overlap the red/blue curves. Note that the reflected (red) wave returns to the left end just in phase with the driver (or incident blue wave), setting up a standing wave with one-half the wavelength just fitting along the string.
employing the same trigonometric identity that we used to get Equation (10.16), we have for the sum

\[ y = y_1 + y_2 = 2A \sin kx \cos \omega t. \]  

\[ (10.17) \]

What is striking about this result is that the sum of these two traveling waves is no longer a traveling wave. At any value of \( x \) the amplitude oscillates at angular frequency \( \omega \), but there is no waveform that travels along the string. In fact there are periodic positions along the string (corresponding to \( kx \) equal to either 0 or multiples of \( \pi \)) where the amplitude is always equal to zero. This type of wave is known as a standing wave.

Because the string length and wave velocity are fixed, for most continuous oscillation frequencies the waves traveling to the right and left will have no particular phase relation, with the wave returning to the left end at different values of transverse displacement at the start of each of the forced oscillations. The result of such a situation will be a net destructive interference and no sustained displacement of the string. Only for a particular set of frequencies, called the resonant frequencies, will standing waves be produced. The lowest possible resonant frequency is called the fundamental frequency, or first harmonic, and is the situation shown in Figure 10.17 (left) in which half of a wavelength fits on the string. The wavelength is then equal to \( \lambda = 2L \), so that the fundamental frequency is equal to \( v_{\text{wave}}/(2L) \), or the inverse of the round-trip time.

As the frequency is increased beyond the fundamental, there will be a sequence of discrete frequencies, called harmonics, at which resonance will occur. At the second harmonic frequency, for example, the wave will reach the right end and reflect back in the same round-trip time but now corresponding to two complete oscillations, so that the resonant frequency is twice that of the fundamental. The wavelength is then equal to \( \lambda = L \), with the second harmonic frequency given by \( v_{\text{wave}}/L \), precisely twice the fundamental frequency. In this case, the waves traveling to the right and left will always produce a point at the center of the string at which there is no displacement. Such a point is called a node and, as can be seen in Figure 10.17, each higher harmonic adds one additional node along the string. The wavelengths of these resonances are given by

\[ \lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots \]  

\[ (10.18) \]

where \( n \) is the harmonic number; \( n = 1 \) refers to the fundamental or first harmonic, \( n = 2 \) to the second harmonic, and so on. The corresponding resonant frequencies are given by

\[ f_n = \frac{v_{\text{wave}}}{\lambda_n} = nf_1. \]  

\[ (10.19) \]

The fourth harmonic is shown in Figure 10.17. The second and higher harmonics are also known as the overtones, with the second harmonic also called the first overtone, the third harmonic also called the second overtone, and so on.
Standing waves on a string are one example of the more general phenomenon of resonance, introduced in Section 1 for the case of simple harmonic motion. In general, resonance is the addition of energy to a system at one of the natural frequencies of the system. In the case of the string, standing waves occur if the driving force frequency is equal to the fundamental or any harmonic frequency of the system, as determined by the length and mass per unit length of the string as well as its tension. As the driving frequency is tuned, a series of resonances with different amplitudes is produced (Figure 10.18).

Standing waves can be set up in any object that is made to vibrate, including all musical instruments at sound frequencies, and bridges, buildings, and other manmade constructions, as well as ocean water at subsonic frequencies (Figure 10.19). We study some of these in connection with sound a bit further in the next chapter. Resonance can occur in many other types of systems including atomic or molecular systems. In these cases involving the microscopic world, electromagnetic oscillations, comparable to mechanical or sound vibrations, produce the resonance. Nuclear magnetic resonance (NMR—the basis for MRI—magnetic resonance imaging) occurs when electromagnetic radio waves are tuned to have the energy needed to produce spin flips in the nuclei and are studied later in this book. A variety of other spectroscopic techniques that involve the interactions of various types of electromagnetic radiation with matter can be analyzed using the concept of resonance.

Even the simpler case of resonance in damped forced harmonic motion, as discussed in Section 1, can serve as the basis for analyzing a variety of physical systems ranging from the mechanical pendulum in a grandfather clock, or a child being rhythmically pushed on a playground swing, to electromagnetic and quantum systems in which radiation acts as

Example 10.3 A steel guitar string with a 10 g mass and a total length of 1 m has a length of 70 cm between the two fixed points. If the string is tuned to play an E at 330 Hz, find the tension in the string.

Solution: From the frequency and the fact that the fundamental has a wavelength equal to twice the distance of 0.7 m, we find that the wave velocity must be equal to \( v = \frac{f \lambda}{2} = 330 \times 0.7 = 462 \text{ m/s} \). Then given the mass per unit length of 0.01 kg/m, we can use Equation (10.14) to find the tension. From \( v = \frac{F_T}{mL} \), we can solve for \( F_T \) to find

\[
F_T = v^2 \left( \frac{m}{L} \right) = (462)^2 (0.01) = 2130 \text{ N}
\]

Enormous tensions are needed in stringed instruments. Steel, nylon, or natural fibrous materials such as catgut are used to support these tensions.

![FIGURE 10.18 Multiple resonances in a real system (such as a string tied at one end) will occur as the driving frequency is varied.](image)

![FIGURE 10.19 Standing wave sand markers where the ocean and a stream meet.](image)
the driving force and the damped oscillations are those of electrons or nuclei in molecules. Just as we saw in Chapter 4 (Section 4) that springs are the natural “picture” that we can use to approximate the forces acting near equilibrium, the addition of a damping and a driving force allow for interactions of the spring with both internal forces (the frictional loss of energy) and external forces (the addition of energy to the system). In biological systems, receptors (of sound, light, or specific molecules) usually involve a resonance. For example, in the next chapter on sound we learn about Helmholtz resonance in the ear and the Békésy resonant waves in the cochlea.

**CHAPTER SUMMARY**

In the presence of a damping force proportional (through the damping constant \( b \)) to velocity, the position as a function of time of a mass \( m \) attached to a spring with spring constant \( k \) is given by

\[
x(t) = (A e^{-\frac{bt}{2m}}) \cos(\omega_{\text{damp}}t), \tag{10.4}
\]

where the angular frequency is

\[
\omega_{\text{damp}} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \tag{10.5}
\]

When the \( b \) is equal to zero, these equations reduce to the simple harmonic motion case:

\[
x(t) = A \cos(\omega_0 t), \tag{10.1}
\]

with the natural frequency \( \omega_0 = \sqrt{\frac{k}{m}} \).

When the oscillator is driven by an external force \( F \) oscillating at \( \omega_{\text{ext}} \) the position of the oscillator is given by

\[
x(t) = A(\omega_0, \omega_{\text{ext}}) \cos(\omega_{\text{ext}} t + \varphi), \tag{10.8}
\]

\[
A = \sqrt{\frac{F/m}{(\omega_0^2 - \omega_{\text{ext}}^2)^2 + \left(\frac{b\omega_{\text{ext}}}{m}\right)^2}}. \tag{10.9}
\]

Such an oscillator exhibits the phenomenon of resonance: the amplitude rises rapidly as the external frequency approaches the natural frequency of the spring and more external energy is able to be absorbed by the system. This model of the driven damped harmonic oscillator using a spring is broadly applicable to a variety of other types of systems.

All periodic waves (with wavelength \( \lambda \) and frequency \( f \)) travel at a speed given by

\[
v = \lambda f. \tag{10.10}
\]

Periodic traveling waves (e.g., waves on a string) can be written showing their displacement as a function of both position \( x \) and time \( t \):

\[
y(x, t) = A \sin(kx - \omega t), \tag{10.13}
\]

where \( k \) is the wave number,

\[
k = \frac{2\pi}{\lambda}. \tag{10.12}
\]

Waves obey the principle of superposition: they pass through each other undisturbed and where they overlap in space, the net amplitude of the wave is equal to the algebraic sum of those of the overlapping individual waves. Interference is a consequence of superposition. Two waves traveling along a string with equal amplitude, wavelength, and frequency but with a phase difference \( \varphi \) between them superimpose to yield a net traveling wave that has an amplitude given by the term below in square brackets and is shifted in phase by \( \varphi/2 \) from either original wave:

\[
y = [2A \cos \frac{1}{2} \varphi \ \sin (kx - \omega t + \frac{1}{2} \varphi)]. \tag{10.16}
\]

In the special case of two overlapping waves of equal amplitude traveling in opposite directions along a string of length \( L \) (perhaps from reflections at the ends), standing waves can be produced:

\[
y = y_1 + y_2 = 2 A \sin kx \cos \omega t. \tag{10.17}
\]

These only occur when the frequency (or wavelength) satisfies the resonance conditions:

\[
f_n = \frac{v_{\text{wave}}}{\lambda_n} = nf_1, \tag{10.19}
\]

\[
\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots \tag{10.18}
\]
QUESTIONS
1. Give several examples of everyday phenomena that approximate harmonic motion. In each case name the source of damping.
2. What are a few examples of forced harmonic motion?
3. Carefully define the amplitude, phase angle, driving frequency, and natural frequency for driven harmonic motion.
4. Name some examples of resonance phenomena, giving the approximate resonant frequency involved.
5. What is the difference between equilibrium and steady state? Which one requires an input of energy?
6. Define wavefront. What is the wavefront shape of each of the following?
   (a) A three-dimensional wave emanating from a point in all directions
   (b) An in-phase wave traveling along the x-direction
   (c) A two-dimensional wave (such as on a drum membrane or the surface of a lake) emanating from a point
7. What is the difference between transverse and longitudinal vibrations of a spring? Distinguish between the wave velocity and spring velocity in each case.
8. What distinguishes a harmonic wave from any other type of wave?
9. Equation (10.13) defines a wave traveling along the positive x-axis, because as time increases x must increase for points of constant phase. How would you write an expression for a wave traveling along the negative x-direction?
10. Why is the potential energy of a stretched string zero at the amplitudes of a traveling wave and maximum at the zero-crossings?
11. Two waves, each of amplitude \( A \) with intensities proportional to \( A^2 \), overlap in space producing an interference effect. Although the total intensity of the two separate waves is proportional to \( 2A^2 \), the net amplitude where they overlap can range from 0 to \( 2A \), so that the net intensity can range from 0 to \( 4A^2 \). Discuss this in terms of conservation of energy.
12. Discuss in words how a node is produced for a standing wave on a string. How does the string move at an antinode?
13. A string of length \( L \), mass per unit length \( \mu \), and tension \( F_T \) is vibrating at its fundamental frequency. Describe the effect that each of the following conditions has on the fundamental frequency.
   (a) The length of the string is doubled with all other factors constant.
   (b) The mass per unit length is doubled with all other factors constant.
   (c) The tension is halved with all other factors constant.
14. When two different stringed instruments play the same fundamental note, what is it that allows you to distinguish the tone from the two instruments, for example, a violin and a viola?
15. Two strings are tied together in a knot. One string has a length \( L \) and a mass \( m \), and the other one has half the length and twice the mass. If the strings are stretched taut and put under tension and a transverse wave travels down the longer string, through the knot into the shorter string, what is the ratio of the wave speeds in the shorter to longer string? What is the ratio of the frequencies in the two strings? The wavelengths?
16. Consider the same two strings tied together as in the previous question. If a positive wave pulse (above the axis of the strings) is sent down the longer string, what will be the polarity of the pulse reflected at the knot? If a positive pulse is sent the other way along the string, what will be the polarity of the reflected pulse in this case?
17. When a snowstorm occurs, often there is a variation in the amount of snow on an electric high-voltage wire strung between two support poles as a result of standing waves. Show what you might expect the pattern to look like for the lowest resonant modes.

MULTIPLE CHOICE QUESTIONS
1. Which of the following is not true of simple harmonic motion of a mass on a spring? (a) The maximum acceleration occurs at the amplitude of motion, (b) the resonant frequency is proportional to the square root of the mass, (c) the period is independent of the amplitude of the motion, or (d) the kinetic and potential energies of the mass exchange with each other at twice the resonant frequency.
2. A disturbance in a string has a node at \( x = 0 \) m, at \( t = 0 \) s. At \( t = 1 \) s, the same node is observed to be at \( x = 5 \) m. This disturbance must be (a) a wave traveling in the negative x-direction with speed 5 m/s, (b) a wave traveling in the positive x-direction with speed 5 m/s, (c) a standing wave with nodes separated by 5 m, (d) either a standing or traveling wave with frequency equal to 1 Hz.
3. A transverse sinusoidal wave travels along a string with a constant speed 10 m/s. The acceleration of a small lump of mass on the string (a) varies sinusoidally in time in a direction perpendicular to the string, (b) varies sinusoidally in time in a direction parallel to the string, (c) is \( 10 \) m/s\(^2\), (d) is zero.
4. In a periodic transverse wave on a string the value of the wave speed depends on (a) amplitude, (b) wavelength, (c) frequency, (d) none of the choices (a)–(c).
5. Two strings are held under the same tension. String A has a mass per unit length that is two times that of string B. The wave speed in A is (a) the same as in B, (b) one half that in B, (c) two times that in B, (d) none of the above.
6. Suppose the tension in a string is given by \( T \) and the mass per unit length by \( \mu \). What are the fundamental dimensions (i.e., \( M \), \( L \), and \( T \)) of the quantity \( \sqrt{\frac{T}{\mu}} \)? (a) \( LT^{-1} \), (b) \( MLT^{-2} \), (b) \( L^2T^{-2} \), (d) \( L^{1/2}M^{-1/2}T^{-1/2} \).
Questions 7–9 refer to: A transverse traveling wave on a string is described by the mathematical expression \( y = (0.10)\sin(2\pi x + 10\pi t) \), where \( x \) and \( y \) are measured in meters and \( t \) is measured in seconds.

7. The frequency of this wave is (a) 10 Hz, (b) 5 Hz, (c) 2 Hz, (d) 1 Hz.

8. This wave is traveling in which direction? (a) +y, (b) −y, (c) +x, (d) −x.

9. The speed with which this wave travels is (a) 1 m/s, (b) 2 m/s, (c) 5 m/s, (d) 10 m/s.

10. Given the traveling wave \( y(x, t) = 0.1 \sin(\pi x - \pi t/2 + \pi/2) \), with \( x \) and \( y \) in meters and \( t \) in seconds, its frequency is (in Hz) (a) 0.25, (b) 2.0, (c) \( \pi \), (d) \( \pi/2 \), (e) none of the above.

11. Antinodes and nodes occur (a) in standing waves, (b) in traveling waves, (c) during beats, (d) in longitudinal waves, (e) none of the above.

12. When a string tied down at both ends is plucked, the resonant frequencies are characterized by all of the following except (a) there must be nodes at both ends, (b) they must satisfy the equation \( f = v_{\text{wave}}/\lambda \), (c) the fundamental frequency is the lowest allowed resonant frequency, (d) the fundamental wavelength is \( L \).

13. Two identical masses are each attached to a spring. The springs are also identical. The masses are driven by the same periodic external force and the response curves (amplitude versus driving frequency) are shown to the right. Which of the following best describes what is seen in the graphs? (a) Mass B is not as well attached to its spring as is mass A. (b) Mass A is at resonance but mass B is not. (c) Mass A experiences more friction than mass B. (d) Mass A experiences less friction than does mass B.

14. Four masses capable of moving along a line are interconnected by springs. These masses are driven into resonance by an external force. The following graphs show the masses’ displacement from equilibrium, at a given instant, in each of the allowed resonant modes. (The end masses in these graphs don’t move; they’re not part of the system.) Rank order the frequencies associated with each graph. (Hint: connect the dots.) (a) \( A > B > C > D \), (b) \( D > C > B > A \), (c) \( A > D > C > B \), (d) \( D > A > B > C \).

15. When two identical harmonic waves of amplitude \( A \) interfere, the net result can be all but which of the following: (a) no wave, (b) a harmonic wave with an amplitude of \( 2A \), (c) a harmonic wave with an amplitude of \( 1.5A \), (d) a harmonic wave with an amplitude of \( 4A \).

16. In forced harmonic motion, as the frequency of the external oscillation driving force approaches the natural frequency of oscillation in a phenomenon called resonance, which of the following occurs? (a) The
period becomes increasingly long. (b) The amplitude becomes increasingly large. (c) The frictional damping becomes increasingly large. (d) The energy steadily decreases.

17. You observe a string under tension (fixed at one end and supporting a hanging weight at the other) to form a standing wave when the driving frequency is 40 Hz. If you replace the 200 g hanging weight with a 100 g weight (but don’t change the wire length) the standing wave with the same shape will occur at about what frequency? (a) 40 Hz, (b) less than 40 Hz, (c) greater than 40 Hz, (d) you can’t form a standing wave with the same shape under these conditions.

18. You are told that the mass per unit length of a wire is $1 \times 10^{-3}$ kg/m and that a 0.1 kg mass is to be used to stretch the wire, by hanging from one end with the other end held fixed. Which of the following is true about the wave speed in the wire? The wave speed (a) depends on the length of the wire, (b) depends on the frequency with which the wire is vibrated, (c) is approximately 1000 m/s, (d) is approximately 30 m/s.

19. The fundamental standing wave on a string of length 1 m that is fixed at both ends vibrates at a frequency of 300 Hz. The speed of waves on this string must be (a) 100 m/s, (b) 150 m/s, (c) 300 m/s, (d) 600 m/s.

20. Suppose a vibrating wire is exactly 1 m long. The standing wave corresponding to the third harmonic on this wire has a frequency of 30 Hz. The wave speed of a transverse wave on this wire (a) is 10 m/s, (b) is 20 m/s, (c) is 30 m/s, (d) cannot be determined from the information given.

21. In restringing a violin A string (fundamental $f = 440$ Hz), if a string with twice the mass/length is incorrectly used and the tension is adjusted to play the correct fundamental, by what factor is the tension different from what it should be using the correct string: (a) $1/2$, (b) $\sqrt{2}$, (c) 2, (d) 4.

22. A car travels over a dirt road that contains a series of equally spaced bumps (a so-called “washboard” road). While traveling at a given speed the driver experiences a very jarring ride. When the driver drives at a higher speed, however, the ride gets smoother. That is because (a) the car actually leaves the ground at higher speeds, (b) the faster moving car actually crushes the bumps and makes the road smoother, (c) the car’s shock absorbers have more friction at higher speeds, (d) going faster in the car forces the suspension to oscillate at a frequency higher than its natural frequency.

23. A damped driven oscillator has an equation of motion given by $ma = -kx - bv + F_0 \cos(\omega_f t)$, where $\omega_f$ is the angular frequency of the driving force. At resonance $ma$ must equal (a) $-kx$, (b) $-bv$, (c) $+F_0 \cos(\omega_f t)$, (d) zero.

**PROBLEMS**

1. You are watching a mass oscillate on a spring. You measure the period to be a constant 1.1 s but you see that the 10 cm initial amplitude of the oscillation halves after 10 s. Write an expression for the time-dependence of the position of the mass $x(t)$ in terms of $t$ with all other factors given as numbers.

2. In the previous problem, how long will it take the mass to lose half its initial energy?

3. A 0.5 kg mass attached to a linear spring, with spring constant 5 N/m and damping constant 0.2 kg/s, is initially displaced 10 cm from equilibrium. (a) What is the natural frequency of oscillation? (b) What is its period of oscillation? (c) How long does it take for the amplitude to decrease to 10% of its starting value? (d) How many oscillations have occurred in this time? (e) What fraction of the initial energy remains after this time?

4. A 0.2 kg mass is attached to a vertical hanging spring, stretching it by 10 cm. The mass is then pulled down an additional 10 cm and released. It is found that the amplitude decreases to 5 cm in 30 s. (a) What is the spring constant? (b) Find the natural frequency of oscillation. (c) What is the damping constant of the spring? (d) Write the equation of motion for the mass as a function of time. (e) Write an equation for the energy of the mass as a function of time.

5. A 1 kg mass is attached to a vertically hanging spring with spring constant 10 N/m and damping constant 0.1 kg/s. Suppose a harmonic driving force with fixed amplitude of 1 N and variable frequency is applied to the mass. Construct the resonance curve showing the amplitude of oscillation as a function of the driving frequency near the natural frequency of oscillation of the mass. Use a set of about 10 points to show the main features of the curve.

6. A vertical spring with a spring constant of 8 N/m and damping constant of 0.05 kg/s has a 2 kg mass suspended from it. A harmonic driving force given by $F = 2 \cos(1.5t)$ is applied to the mass. (a) What is the natural angular frequency of oscillation of the mass? (b) What is the amplitude of the oscillations at steady state? (c) Does this amplitude decrease with time due to the damping? Why or why not?

7. A 4 m long rope weighing 1.4 N is stretched so that the tension is 10 N. The left end is then made to oscillate vertically at 4 Hz by shaking the rope up and down a total distance of 10 cm. (a) What is the speed of the traveling waves on the rope?
(b) What is the wavelength of the waves?
(c) Write the equation of the traveling waves along the rope (ignoring the reflected waves from the far end).

8. A traveling wave on a string is described by the equation $y(x,t) = 0.1 \sin(25x - 500t)$.
(a) What are the wavelength, frequency, and amplitude of the wave?
(b) What is the wave velocity?
(c) If the mass density of the string is 0.001 kg/m, find the tension in the string.

9. A wave traveling on an elastic string has a 5 cm amplitude, a 25 cm wavelength, and a period of 0.01 s.
(a) Write an equation for the traveling wave $y(x,t)$ traveling in the positive $x$-direction.
(b) Find the wave speed.
(c) If the string is under a tension of 10 N, find the mass density of the string.

10. A 10 m elastic cord with a mass of 0.42 kg has its left end tied to a wall and is pulled with a force of 50 N at the right end. When the right end is vibrated vertically according to the equation $y = 0.04 \sin(2.5t)$, where $y$ is in meters and $t$ in seconds, write the equation for the wave traveling to the left.

11. A string is tied at one end to a fixed point and the other is attached to a 1 kg weight after passing over a frictionless pulley. The 4 m long string weighs 0.1 kg and the distance between the fixed point and the pulley is 3.5 m.
(a) Find the speed of transverse waves on the string.
(b) What is the fundamental frequency?
(c) What is the wavelength of the fourth harmonic?

12. Derive Equation (10.16) for the superposition of two equal amplitude traveling waves with a phase difference $\phi$ between them.

13. Two traveling waves with the same amplitude $A$, frequency $f$, and wavelength $\lambda$, but out of phase with each other by one quarter of a wavelength, are both traveling to the right and superpose in space. Find the amplitude, wavelength, and frequency of the resulting wave in terms of the given symbols. Write the equation of the resulting traveling wave $y(x, t)$.

14. A sinusoidal wave is traveling at 300 m/s along a string with a mass-per-unit-length of 0.002 kg/m. If the wave has an amplitude of 0.01 m and a wavelength of 0.05 m find the following.
(a) The equation for the traveling wave, $y(x,t)$
(b) The tension in the string
If a second identical traveling wave is on the same string but is shifted by 45° with respect to the first, find
(c) The net amplitude where the two waves overlap on the string
(d) The equation for the net traveling wave, $y(x,t)$

15. Standing waves are set up on a 1.5 m long string under tension and fixed at both ends. If the distance between nodes along the string is 0.25 m what is the wavelength of this mode and what harmonic is it?

16. A 3 m long string with a mass-per-unit-length of 0.005 kg/m is tied down at one end and has a 5 kg mass hanging over a pulley from the other end of the string putting it under tension. If standing waves are set up, find the frequency of the fundamental mode and of the fourth harmonic.

17. According to the Guinness book, the world’s largest double bass instrument was 14 feet tall and had 4 strings (of equal length) totaling 104 feet in length. If the heaviest of these strings had a mass of 2 kg, find its fundamental frequency when under a tension of 5000 N. This sound would be felt but not heard.