

## ‘Operational’ energy conditions

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**Abstract.** The negative energy densities predicted by relativistic quantum field theories are investigated. Combining the quantum inequalities of Ford and Roman, which restrict the time for which a negative energy density may persist, with quantum limitations on measuring devices, it is shown that a Klein–Gordon field in Minkowski space obeys the *operational weak energy condition*: the energy of an isolated device constructed to measure or trap the energy in a region, plus the energy it measures or traps, cannot be negative. There are good reasons for thinking that a parallel result holds locally for linear fields in curved spacetimes. Indeed, there is reason to think an operational dominant energy condition is satisfied.

A specific thought experiment to measure energy density is analysed in some detail, and the operational positivity is clearly manifested.

If such operational energy conditions are valid generally, a case can be made that spacetime cannot be adequately modelled classically when negative energy density effects are significant.

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### 1. Introduction

A startling prediction of relativistic quantum field theory is that, while the total energy of a system should be positive or zero, the energy density, and hence the energy of a subsystem, can be negative (Epstein *et al* 1965). And indeed this possibility is present generically. Even for a Klein–Gordon field on Minkowski space, for any smooth compactly supported bump function  $B$ , the expectation values

$$\left\langle \int \widehat{T}_{00} B \, d^3x \right\rangle, \quad (1)$$

for  $\widehat{T}_{00}$  the renormalized Hamiltonian density, are unbounded below (Helfer 1996). Thus, states with arbitrarily negative energy densities are always available. The set of states with this expectation value equal to  $-\infty$  is dense in the Hilbert space.

Negative total energy densities have never been directly observed. Although the Casimir effect is sometimes cited as a case where a negative energy density has been measured, in Casimir-type experiments it is one of the ‘pressure’ components of the stress–energy operator ( $\widehat{T}_{zz}$ ) which is measured, and not the energy density ( $\widehat{T}_{tt} = \widehat{T}_{00}$ ). There is an indirect mathematical relation between them (the long-time average of the force can be computed as minus the gradient of the energy), but they are different quantum observables. Just such distinctions will be central in this paper. The emphasis will be on the question of what direct local measurements of  $\widehat{T}_{tt}$  are possible.

Nevertheless, negative total energy densities have received extensive theoretical investigation because they contravene a basic tenet of classical physics. Indeed, if there were no restrictions on the negative energies achievable, there would be gross macroscopic consequences: an ordinary particle could absorb a negative energy and become a tachyon; an isolated patch of negative energy would give rise to a repulsive gravitational field; one could violate the second law of thermodynamics by using negative energies to cool systems without an increase of entropy (Ford 1978, 1991, Davies 1982, Grove 1988); and the general-relativistic effects might include traversable wormholes, ‘warp drives’ and time machines (Morris and Thorne 1988, Morris *et al* 1988, Alcubierre 1994, Everett 1996, Ford and Roman 1996, Pfenning and Ford 1997). In addition, it is something of a puzzle why such states do not interfere with the dynamics of the quantum fields: why do not perturbations (which are always present) send the field cascading through these negative-energy states, with a corresponding release of positive-energy radiation? It is a matter of common experience that such effects do not occur, or at least not often, and therefore there must be some mechanism restricting the production of negative energy densities, their magnitudes, durations or interactions with other matter.

Such a mechanism was first proposed by Ford (1978), and has been investigated by him and Roman (Ford 1991, Ford and Roman 1997 and references therein). They establish restrictions on the negative energy density and flux that can persist for a given time, and present arguments that (for free Bose fields in Minkowski space) the negative energy  $-\Delta E$  localizable in a time of order  $\Delta t$  should satisfy a *quantum inequality*

$$\Delta E \Delta t \lesssim \hbar. \quad (2)$$

These inequalities are powerful; they evidently limit the occurrence of negative energies considerably. However, they do not as they stand seem to be a full explanation. For one thing, the argument for (2) depends on a certain ‘coherence’ assumption, which is not generally valid<sup>†</sup>. It is also not clear that simply restricting the occurrences of large negative energies to short times is enough to rule out unphysical effects. Indeed, explicit analyses of attempts to violate the second law of thermodynamics indicate that while the quantum inequalities play a key role, an equally important one is played by limitations on the measuring devices. (See Ford 1978, 1991, Davies 1982, Grove 1988. These papers also point out that identifying the characteristic time  $\Delta t$  which is relevant for a particular physical problem may not be an easy matter.)

In section 2 of this paper, I shall re-examine the derivation of the quantum inequalities, find a way to repair the ‘coherence’ condition, and argue further that a device constructed to measure or capture a local negative energy  $-\Delta E$  must itself have energy of at least  $\Delta E$ . One may say briefly that *operationally the total energy must be non-negative*. In section 3, I shall extend the reasoning to establish an operational dominant energy condition for free fields in Minkowski space. This says, roughly, that the total 4-momentum density of the field plus the measuring device must be future-pointing.

In section 4, I shall describe a thought device which measures energy density, and discuss its quantum limitations. For this device, I find even more stringent restrictions than the operational energy conditions. The Planck scale appears explicitly as a limitation on its measurements, and below the Planck scale timing errors prevent one from approaching the regime where the negative energy densities are comparable to the energy density of the

<sup>†</sup> Ford (1991) recognized that (2) could be violated if the coherence condition does not hold. A weaker bound he established (our inequality (4)), which is suggestive of the averaged weak energy condition in general relativity, does not depend on this condition. This weaker condition serves to severely limit wormholes and ‘warp drives’ (Ford and Roman 1996, Pfenning and Ford 1997).

device (see Ford *et al* 1992, Ford and Roman 1993 for previous discussions of detectors).

In section 5, I shall show that at least in two limiting regimes, in a thought experiment using this device to measure the energy density between the plates of a Casimir apparatus, the operational positivity of energy still holds. (Other regimes would require better quantum inequalities on the Casimir configuration than are presently known.) This is remarkable, as the Ford–Roman inequalities do not apply on this scale, and indeed treat negative energy densities which arise in a way distinct from the negative Casimir energy.

Section 6 gives a more extensive discussion of what operational positivity of energy means, assesses the likelihood that it holds universally, and analyses its implications for gravity. One can make a serious argument that spacetime cannot be adequately modelled by a classical manifold with classical metric in regimes where negative energy densities are significant.

The final section summarizes the main conclusions.

The metric signature used is +---.

## 2. The quantum inequalities

Ford and Roman have given several derivations of the quantum inequalities, but the elements which are relevant here are common to all. Consider the quantity

$$\mathcal{E} = - \inf_{|\Phi\rangle} \left\langle \Phi \left| \int_{-\infty}^{\infty} \widehat{T}_{00}(t, 0, 0, 0) b(t) dt \right| \Phi \right\rangle, \quad (3)$$

(over normalized states  $|\Phi\rangle$  in the Fock space of a Klein–Gordon field in Minkowski space), where  $b(t) = t_0/[\pi(t^2 + t_0^2)]$  is a 'sampling function' with integral unity and characteristic width  $t_0$ †. It is shown that

$$\mathcal{E} \leq k\hbar c (ct_0)^{-4}, \quad (4)$$

where it is known that the numerical constant  $k \leq 3/(32\pi^2)$ . Up to this point the argument is essentially mathematical.

The next step is physical. If a device were to be constructed to measure or trap this negative energy within an interval of length  $t_0$ , then, in order to function coherently, the linear dimension of the device must be no larger than  $ct_0$ ‡. Thus the magnitude of the negative energy within the device is

$$\Delta E \leq \mathcal{E} (4\pi/3) (ct_0)^3 \leq (4\pi k/3) \hbar t_0^{-1} \leq \hbar / (8\pi t_0). \quad (5)$$

This hypothesis of coherent functioning deserves closer scrutiny.

The trouble here is that although it may be reasonable to think of an experiment as a whole (including preparation at the start and collection of data at the end) as 'coherent' on a time scale  $T_0$ , the scales  $t_0$  of the components of the experiment may be much smaller. For example, suppose we had  $N$  devices obeying (5), so capable of detecting or trapping a negative energy  $-\varepsilon(4\pi k/3)\hbar t_0^{-1}$  in time  $t_0$ ; here  $\varepsilon < 1$  is the efficiency of the device. These devices are arranged in an array in space, and in a common rest-frame. Each carries

† The sampling function is peaked over an interval of characteristic width  $\sim t_0$ , but not supported only there. It is not possible to localize to a sharply demarcated time interval, because the quantity  $\langle \Phi | \widehat{T}_{00}(t, 0, 0, 0) | \Phi \rangle$  is a distribution, and turns out to involve terms like  $\delta'(t)$ . It is this which is behind difficulties in identifying the correct  $\Delta t$  for physical applications.

‡ Perhaps one should use  $\frac{1}{2}ct_0$  here. This would make the force of our later arguments somewhat stronger.

a clock which has been synchronized with (say) a master clock at the centre of the array. At a preset time, each device operates. Then, if the field is in a suitable configuration, the total negative energy absorbed will be  $-N\varepsilon(4\pi k/3)\hbar t_0^{-1}$  and the interval will be  $t_0$ . (Note that there is no requirement that the devices be near one another, so they can be separated far enough apart that locality considerations guarantee that the quantum field can indeed be in a state which will produce such a negative energy. Also note that while it is true that construction of the array of devices requires a time scale different than  $t_0$ , that time scale is *larger*, namely the time required to synchronize the devices, greater than  $\sim N^{1/3}t_0$ .) By choosing  $N$  large enough, we can arrange for an arbitrarily large negative energy to be trapped within a time  $t_0$ . Thus even if we start from ‘coherent’ devices, we can create others which violate the quantum inequality (5).

We can repair this by taking into account the energy of the measuring device. A device which measures  $\int_{-\infty}^{\infty} \widehat{T}_{00}(t, 0, 0, 0)b(t) dt$  must involve some sort of clockwork mechanism and transducer, which function to weight the contributions of  $\widehat{T}_{ab}$  at different times by  $b(t)$ . This clockwork must be able to resolve time increments of order  $t_0$ . (Actually, in order to treat the function  $b(t)$  as free from quantum indeterminacy, the temporal resolution must be finer.) Now let us recall that a clock mechanism which is accurate to a time of order  $\Delta t$  must have mass  $\gtrsim \hbar/(c^2 \Delta t)$  and so energy  $E_{\text{mech}} \gtrsim \hbar/\Delta t$  (Salecker and Wigner 1958). For any one clock having energy  $E_{\text{mech}}$ , then, controlling a measuring device, the inequality (5) applies with  $t_0 \sim \hbar/E_{\text{mech}}$ .

This suggests that any device controlled by a clock of energy  $E_{\text{mech}}$  can detect or trap negative energies  $-\Delta E$  with  $-\Delta E + E_{\text{mech}} \geq 0$  only. It should be made clear that this argument is not a mathematical proof. For one thing, the quantities  $\Delta t$  and  $E_{\text{mech}}$  are only defined as orders of magnitude, and it is in this sense that  $E_{\text{mech}} \gtrsim \hbar/\Delta t$  is known to hold. For another, the quantum inequality (5) has only been established for one form of sampling function. Nevertheless, the numerical factor  $1/8\pi$  in inequality (5) is far enough below unity that it strongly suggests  $E_{\text{mech}} \geq \Delta E$ . For the remainder of this paper, I shall assume this is the case.

With this assumption, notice that a collection of measuring or trapping devices deployed and set to function simultaneously (or, more generally, at spacelike separations), as in the example above, will also have total energy in excess of the negative energy it can detect or trap.

One may summarize this contention by saying that *operationally, the energy must be non-negative*, that is, the sum of the measured energy and the energy of the measuring device must be non-negative.

### 3. The dominant energy condition

The treatment so far concerns the energy of a finite system, as measured by an inertial observer. The result localizes: even if one tries to separate the clockwork used for measuring  $\int_{-\infty}^{\infty} \widehat{T}_{00}(t, 0, 0, 0)b(t) dt$  from the world-line  $(t, 0, 0, 0)$ , one must still transmit timing signals to the vicinity of this world-line, and these signals must resolve times  $\lesssim t_0$ , which means the quanta carrying the signals must have energies  $\gtrsim \hbar/t_0$ . Thus locally the total energy, of the field plus the measuring device, must be non-negative. One may call this the *operational weak energy condition*:  $T_{ab}^{\text{op}} t^a t^b \geq 0$  for all timelike vectors  $t^a$ .

It is possible to derive a stronger result, the *operational dominant energy condition*:  $T_{ab}^{\text{op}} t^a u^b \geq 0$  for all future-pointing vectors  $t^a$  and  $u^a$ . The changes needed to the treatment above are as follows.

3.1. A quantum inequality for momentum density

Let

$$\Pi_a = \left\langle \int_{-\infty}^{\infty} \widehat{T}_{a0}(t, 0, 0, 0) b(t) dt \right\rangle. \quad (6)$$

Then one can prove  $\Pi_a u^a \geq -(3/32\pi^2) t_a u^a \hbar c / (ct_0)^4$  for any future-pointing vector  $u^a$ .  
Let

$$P_a = \left\langle \int_{-\infty}^{\infty} dt \int_{\|x\| \leq ct_0} d^3x \widehat{T}_{a0}(t, x) b(t) \right\rangle \quad (7)$$

be the expectation value of the 4-momentum measured in an experiment controlled by a clock resolving times  $\sim t_0^\dagger$ . Then  $P_a u^a \geq -(1/8\pi) t_a u^a \hbar / t_0$ ; equivalently

$$P_a = -(1/8\pi)(\hbar/t_0)t_a + \text{a future-pointing vector}. \quad (8)$$

Only a few modifications to the analysis of Ford and Roman (1997) are needed to establish this, and I shall simply indicate these, in the notation of that paper. The main point is that the lemma of appendix B in Ford and Roman (1997) remains valid when  $p_j^* p_{j'}$  is replaced by any Hermitian matrix  $P_{jj'}$  which has non-negative eigenvalues, since each such matrix is a sum of terms of the form  $p_j^* p_{j'}$ . It holds in particular when  $j, j'$  are replaced by the wavevectors  $\mathbf{k}, \mathbf{k}'$  and  $P_{\mathbf{k}\mathbf{k}'} = (2u_{(a} t_b) k^a k'^b - u_a t^a k_b k'^b) / \sqrt{\omega\omega'}$ . The rest of the changes are self-evident.

3.2. Constraints on the measuring device

Consider a clock which may be boosted relative to  $t^a$ . If the clock is required to have resolution  $\Delta t$  in the  $t^a$ -frame, then its resolution in its own frame must be  $\Delta t/\gamma$ , with  $\gamma$  the usual Lorentz factor. Its mass must be

$$m \gtrsim \hbar\gamma / (c^2 \Delta t). \quad (9)$$

Let the clock's 4-momentum  $P_a^{\text{clock}}$  be  $(E, p)$  in the  $t^a$ -frame, so  $E = mc^2\gamma$  and  $p = mc\beta\gamma$ . Then

$$E^2 - mc^2 E/\gamma - p^2 c^2 = 0 \quad (10)$$

from which

$$(E - mc^2/(2\gamma))^2 - p^2 c^2 = m^2 c^4 / (4\gamma^2). \quad (11)$$

This means

$$P_a^{\text{clock}} = mc^2/(2\gamma)t_a + \pi_a, \quad (12)$$

where  $mc^2/(2\gamma) \gtrsim \hbar/(2\Delta t)$  and  $\pi_a$  is timelike future-pointing with  $\pi_a \pi^a = (mc^2/2\gamma)^2 \gtrsim (\hbar/2\Delta t)^2$ .

† Inclusion of a spatial bump function in the integral defining  $P_a$  is possible, and does not affect the argument.

### 3.3. The operational dominant energy condition

Combining the results of the two previous subsections, we see that for any future-pointing vector  $u^a$ , the sum of the expectation value  $P_a u^a$  of the  $u^a$ -component of the momentum and the corresponding component of the momentum of the clock which controls the sampling satisfies

$$(P_a + P_a^{\text{clock}})u^a \geq mc^2/\gamma - (1/8\pi)\hbar/t_0, \quad (13)$$

which we expect to be positive by (9).

A comment about the interpretation of this is in order. Here  $P_a$  is the expectation value of  $\widehat{T}_{ab}t^b$  smeared over a volume in Minkowski space. The components of this smeared operator do not generally commute (one cannot simultaneously measure the components of the 4-momentum in a finite box, because of edge effects). Thus it is perhaps too strong to say that the 4-momentum is operationally future-pointing, since the 4-momentum of the field within a finite box cannot, strictly speaking, be measured. What I have shown is that for any future-pointing vector  $u^a$ , the operator  $u^a P_a^{\text{op}}$  is non-negative, where  $P_a^{\text{op}}$  is the sum of the clock's 4-momentum and the 4-momentum operator for the field in a box.

## 4. A model measurement of energy density

I shall describe here a thought experiment to measure energy density (and thus test the weak energy condition), and examine its quantum limitations. Of course, investigation of any one device cannot prove that there are parallel limitations for all devices; but it does provide a challenge to do better.

The measurement must depend on some coupling to the quantum field, and in order to keep the model as realistic as possible and not make *ad hoc* assumptions about the coupling, I shall consider a device which detects the energy density gravitationally.

Suppose there are several nearby world-lines. One is occupied by an observer, who carries a clock, a photon source, and a photon detector. The other world-lines are those of mirrors. The observer sends out pulses of light to the mirrors, and measures the times it takes them to return. Each such measurement of time provides an estimate of the distance to the mirror. After two measurements (of light bouncing from the same mirror), the observer can estimate a component of the relative velocity of the mirror, and after three measurements she can estimate a relative acceleration and hence one component of the curvature,  $R_{0i0i}$  for a mirror in the  $i$ th direction. By averaging over the mirrors in the three spatial directions, she gets an estimate of  $R_{00}$ . Assuming the stress-energy is trace-free, this measures the energy density.

(Ideally, one would like an experiment which directly measured the energy density, without appealing to the trace-free nature of the stress-energy. Such devices can in principle be constructed by considering four, relatively boosted, apparatuses of the sort described above, and appropriately summing their outputs; similarly one could test the dominant energy condition. A device like this would be subject to requirements at least as strict as those to be considered here, unless one could find some cancellation of errors when summing the outputs. This would be rather complicated and will not be considered further. In any event, the trace of the stress-energy is a c-number for electromagnetism which is second order in the gravitational field.)

A number of factors constrain the design of the device and limit its accuracy. Let the accuracy of the clock be  $\Delta t$ , and the time between bounces to the  $i$ th mirror be  $\sim t_i \geq \Delta t$ . Then:

- (i) The clock's mass satisfies  $mc^2 \gtrsim (\hbar/\Delta t)(t/\Delta t)$ , where  $t$  is its running time,  $t \sim \max\{t_1, t_2, t_3\}$  (Salecker and Wigner 1958).
- (ii) If the extents of the photons' wave packets are not to contribute appreciably to timing errors, the photons' energies must be  $\gtrsim \hbar/\Delta t$ .
- (iii) If the mirrors are not to be significantly accelerated by the photons, each of their rest-energies must be  $\gtrsim \hbar/\Delta t$ . (One could try to use less massive mirrors and correct kinematically for the effects of the photons' impacts, but then the photons are red-shifted after impact and there is a consequent loss of accuracy.)
- (iv) The mirror and the observers must be outside each others' Schwarzschild radii; this means  $ct_i \gtrsim Gm/c^2$ .
- (v) To measure the directions of the mirrors relative to the observer (in order to properly weight the sum  $\sum_{ij} R_{0i0j} g^{ij}$ ) to a given accuracy  $\Delta\theta$  requires an angular momentum  $\gtrsim \hbar/\Delta\theta$ .

One can see how these constraints enforce the operational positivity of energy. The apparatus measures the average energy density over a time  $\sim t$  and spatial extent  $\sim ct$ ; by the Ford–Roman inequalities, the magnitude of the negative energy density in a field is  $|E_{\text{neg}}| \lesssim (1/8\pi)\hbar/t$ . However, the energy of the apparatus must be greater than this, by (i)–(iii).

The full force of the constraints has not yet been used, and indeed they imply more stringent limitations on the total energy density.

First, combining (i) and (iv), one finds

$$\Delta t \gtrsim (G\hbar/c^5)^{1/2} = t_{\text{Planck}} . \tag{14}$$

Thus the Planck scale explicitly limits the measurements. Now we shall show that timing errors keep one from coming close to violating positivity of energy, unless the Planck scale is approached.

The uncertainty in the measured curvature due to timing errors is  $\Delta R \sim \Delta t/(c^2 t^3)$ , leading to an uncertainty in the measured energy of  $\Delta E \sim \Delta R c^7 t^3/G \sim c^5 \Delta t/G$ . Then

$$\Delta E/|E_{\text{neg}}| \gtrsim c^5 t \Delta t/(G\hbar) \gtrsim (c^5/G\hbar)(\Delta t)^2 = (\Delta t/t_{\text{Planck}})^2 . \tag{15}$$

This places severe restrictions on the possibility of observing total energies which are 'close to' negative.

### 5. The Casimir effect

I shall now show that, at least in two limiting regimes, if the device described above were used to measure the Casimir energy density, the operational positivity of energy would still hold. (Investigations beyond these regimes would require better quantum inequalities for the Casimir configuration than are presently known.) This is remarkable because: (i) the Ford–Roman inequalities do not hold in this case; and (ii) the negative Casimir energy density arises in quite a different way from those considered in the Ford–Roman analysis.

When a free field is quantized in a restricted volume with suitable boundary conditions, the stress–energy operator takes the form

$$\widehat{T}_{ab} = \widehat{t}_{ab} + t_{ab}^{\text{Casimir}} , \tag{16}$$

where  $\widehat{t}_{ab}$  is a normal-ordered operator quadratic in the fields, and  $t_{ab}^{\text{Casimir}}$  is a c-number, the Casimir-type contribution. The negative energy-density effects from the two terms are distinct<sup>†</sup>. Those from  $\widehat{t}_{ab}$  are ultraviolet effects which appear when the Hamiltonians in question do not correspond to perfect symmetries of the theory. (For example, measuring the energy in a region corresponds to evolving only the field within the region.) But negative Casimir energies represent a displacement of the vacuum relative to that of Minkowski space. For the original Casimir effect, two plane parallel perfect conductors separated by a distance  $l$  are found to have

$$t_{ab}^{\text{Casimir}} = (\hbar c \pi^2 / 720 l^4) \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{bmatrix} \quad (17)$$

between the plates, in a Cartesian coordinate system adapted to the symmetry of the problem.

With this preamble, we can distinguish the two limiting regimes.

The first is the local one. By this one means a measurement of the average energy density over a spacetime volume of spatial extent  $\sim a$  and temporal extent  $\sim t$  such that  $a, ct \ll l$  and the distance of the volume to either plate is  $\gg a, ct$ . One expects measurements in such regions to be adequately modelled by those of free fields in Minkowski space, and therefore the operational positivity to hold.

The second is the case of long times, that is, average energy-density measurements over times  $\gg l/c$  (and over spatial volumes of some finite size). For sufficiently long times, one expects fluctuations in the field to average out, so the energy density can be well modelled by the c-number term. (There will be spatial fluctuations, too, but over long enough times one expects these to be negligible. See Barton (1991a, b) for an illuminating discussion.)

Start by considering the effect of the clock's mass on the mirror in the  $z$ -direction. The curvature generated by the clock will be  $\sim Gm/(c^5 t_z^3)$ , whereas the Casimir energy density contributes a curvature  $\sim (G/c^4) \hbar c \pi^2 / 720 l^4$ . Here  $t_z$  is the time between photon bounces in the transverse direction. Since  $ct_z \lesssim l$  and  $mc \gtrsim \hbar/l$  (for the clock's Compton wavelength to be smaller than  $l$ ), the field due to the clock will dominate the negative Casimir energy effects. This distortion is not averaged out by summing over the spatial directions, and must be subtracted to get a measurement of the energy density to the required accuracy.

To do the subtraction, one must know  $mc^2/(ct_z)^3$  to an accuracy of better than  $\hbar c \pi^2 / 720 l^4$ , that is,  $\Delta(mc^2/(ct_z)^3) \lesssim \hbar c \pi^2 / 720 l^4$ . In particular, this means  $mc^3(\Delta t_z)/(ct_z)^4 \lesssim \hbar c \pi^2 / 720 l^4$ , from which

$$mc^2 \Delta t_z \lesssim \hbar (ct_z/l)^4 \pi^2 / 720 \lesssim \hbar \pi^2 / 720. \quad (18)$$

However, this would be a contradiction of the fundamental inequality of Salecker and Wigner.

Thus, in both the local and long-time average regimes, one expects operational positivity of energy to hold for the configuration which gives rise to the classical Casimir effect.

## 6. Discussion

### 6.1. What operational positivity means

Operational positivity of energy does not contradict the prediction of negative energy-density states by relativistic quantum field theories. For example, in the Casimir effect, there is

<sup>†</sup> For a link between the two, see the comments on the Casimir effect and quantum electrodynamics in section 6.2.

no dispute that the energy-density operator is *mathematically* not positive: it has a good mathematical definition, and its spectrum includes negative values. Equally, there is no dispute that to compute the energy of the entire Casimir system, of the plates plus the field, the negative binding energy must be accounted for. More generally, there is no dispute that negative energy densities may contribute to lower the energy of an unequivocally observable positive energy object. What operational positivity asserts is that one could never *locally experimentally* verify a *total* negative energy in a region. Put another way, it asserts that certain negative energy-density effects are shrouded in quantum measurement problems.

It is important to keep in mind that it is something of an abbreviation to speak of measurement of energy or energy density in the present context. What is really measured is the integral of  $\hat{T}_{ab}$  against a sampling function. The results of such measurements are not additive, and so the sense in which an energy density exists is less direct than in a classical theory. In a given experiment, one measures an *average* of the stress–energy over a given scale (set by the sampling function), and the same state, measured at different scales, would generally give rise to different values of the energy density. The quantum inequalities lead us to expect that a state might have very negative energy densities when measured on a sufficiently fine scale, but only slightly negative energy densities on gross scales (cf Kuo and Ford 1993).

While the operational positivity condition is a limitation on measurement, it is different from the usual limitations on conjugate observables in three ways:

- (a) It arises not from a failure of commutativity, but from the time–energy uncertainty relation which requires a clock with a given resolution to have a correspondingly great mass. (This should be distinguished from the relation which, for example, relates the widths of spectral lines to the stability of the states.) While this relation is little discussed in texts (perhaps because it has no neat expression in the usual mathematical formalism of quantum theory, where time is only a parameter), its essence goes back at least to the Bohr–Einstein dialogue of 1930 (Bohr 1958).
- (b) The inequality refers not just to the observables within the quantum field theory, but also to the measuring device outside the theory.
- (c) The inequality cannot be expressed as a purely abstract statement about operators on Hilbert space, but refers to the *local measurement* of the stress–energy: it occurs because a clock, or other timing signal, must be present *where* the stress–energy is measured.

To emphasize (c), there are cases in which there is a clear sense in which a non-local measurement violates operational positivity. Consider an experimenter who prepares a Casimir apparatus. It is common in quantum theory to regard the preparation of a state as a measurement (followed by a selection, rejecting those states which are unsuitable). Thus the preparation of the Casimir experiment leaves the system in a state which is (to the required accuracy) perfectly well known. This state is an eigenstate of the operator measuring the energy of the field between the plates, and that energy is negative. However, the measurements done in preparing the state (adjusting the plates, measuring their positions and verifying that they are held steady while the field inside equilibrates) are *not* local to the spacetime region between the plates after the field has equilibrated.

It is an open question whether there are restrictions beyond the purely local in the measurement of energy density. For example, could one infer from the gravitational multipole moments of an object, as measured outside of it, that there was a negative energy density somewhere inside?

At present, the only uncontroversial conclusions to be drawn from this investigation are negative. Any arguments depending on the detection or absorption of negative energies

should be re-examined to see if they are affected by quantum measurement limitations.

However, when a limitation as a matter of principle is discovered, there is the possibility that the physics of the situation is not understood at a deep level. In section 6.3, I shall argue that this is indeed the case, and that spacetime cannot be adequately modelled classically when negative total energy-density effects are significant.

## 6.2. Generality of the results

The operational dominant energy condition has only been established for free fields in Minkowski space. (The investigation of the Casimir effect concerned only the weak energy condition, and concluded only that the specific device in question could not violate positivity of energy—it is possible that some other measurement can be constructed for which there is a violation.) Nevertheless, the results are suggestive enough to make the following conjecture.

*Conjecture 1.* The operational dominant energy condition is universally valid: for any local measurement of the energy–momentum density one has  $T_{ab}^{\text{op}} t^a u^b \geq 0$ , where  $T_{ab}^{\text{op}}$  is the sum of the stress–energy of the field and of the measuring device.

Such a statement cannot be mathematically proved within the present framework of quantum field theory. The remainder of this subsection assesses the evidence for and against the conjecture.

For linear fields on Minkowski space, one would expect to be able to establish Ford-type inequalities, and the operational energy conditions would follow. For nonlinear field theories, one would not expect such simple, universal, Ford-type relations. Still, the apparent validity of operational positivity for the Casimir effect is circumstantial evidence that the operational weak energy condition holds for quantum electrodynamics. The argument is this: presumably, the correct first-principles treatment of the Casimir effect starts with a quantum electrodynamic Lagrangian, including all of the ionic and electronic structure of the conductors. The usual, practical, calculation that is done should be the result of integrating out the dynamically frozen degrees of freedom from this first-principles treatment, and the quantity  $t_{ab}^{\text{Casimir}}$  is the contribution of these frozen degrees of freedom. Thus the operational positivity in the Casimir effect would be a special case of operational positivity in quantum electrodynamics, and the validity of the former is circumstantial evidence for the latter.

The most serious objection to the conjecture is that it could be violated by simply having a sufficiently large number of elementary field species. However, the number required (if they are all free fields) is  $\gtrsim 8\pi$  (from inequality (5)), and it is quite plausible that this is not achieved.

The operational weak or dominant energy conditions would hold for linear fields in curved spacetime if one had a curved-space analogue of the energy-density inequalities (4) or (8). At present, no such general result is known, but it is generally believed that such results should exist. It may be too much to hope for very general results with non-compactly supported sampling functions, but one may reasonably make the conjecture:

*Conjecture 2.* Let  $b(t)$  be a smooth, compactly supported bump function of area unity. Then, for a linear quantum field on a globally hyperbolic spacetime, one has

$$\left\langle \int_{-\infty}^{\infty} \widehat{T}_{ab}(\gamma(t)) \dot{\gamma}^a u^b b(t/t_0) dt/t_0 \right\rangle \geq -C(b) \dot{\gamma}_a u^a t_0^{-4} \quad (19)$$

for small enough  $t_0$ , where  $C(b)$  is positive and depends only on  $b$ , the curve  $\gamma(t)$  is a timelike geodesic parameterized by arc length increasing toward the future and  $u_a$  is a future-pointing timelike vector, parallel-propagated along  $\gamma(t)$ .

While this has not at present been proved even in Minkowski space, there are good if somewhat technical reasons for believing it, and for expecting that  $b$  can be chosen so that  $C(b)$  is small (cf Flanagan 1997, Song 1997). Then the arguments of the previous sections apply, and the operational dominant energy condition holds in curved spacetime.

### 6.3. Implications for gravity

Key to the development of both relativity and quantum theory was what is now called operationalism: the idea that the theory should be formulated in terms of observables. This criterion has been an invaluable guide in the development of quantum field theory in curved spacetime, and should not be ignored. I shall argue here that if we adhere to it, and grant the operational weak energy condition, then spacetime cannot be adequately modelled classically in a regime where negative energy densities are important. While this argument is not conclusive, it is strong enough to take seriously.

That negative total energy densities of themselves preclude a classical model of spacetime is a radical suggestion, since it implies that the limit of validity of classical general relativity may be reached far above the Planck scale. I shall try to make clear exactly the sense in which this limitation is supposed to occur.

Throughout, one must remember that the energy density does not exist as a single operator, but as an operator-valued distribution. Thus the ‘regime’ in question refers not just to a region of spacetime, but also to a scale of measurement set by the sampling function  $f^{ab}$  against which  $\widehat{T}_{ab}$  is integrated. That negative energy densities are important in this regime means that the the projections of the state onto the eigenspaces of  $\int \widehat{T}_{ab} f^{ab} d(\text{vol})$  have significant components with negative eigenvalues. The magnitudes of these components may depend strongly on the scale over which  $f^{ab}$  varies. In an experiment, the function  $f^{ab}$  is determined by the apparatus used.

According to operationalism, we should ascribe a direct classical meaning to a regime if there is, in principle at least, a means of determining its structure by measurements, and that structure is classical. However, the operational weak energy condition, together with Einstein’s field equation, precludes any local, direct, measurement of the geometry in a regime of negative energy density (since the stress–energy can be inferred from the geometry). In order to understand the nature of the limitations involved in more detail, I shall consider attempts to measure the geometry of spacetime which are on the verge of revealing negative energy densities.

I begin by giving more precise statements of when a classical spacetime may be considered an adequate model. This may be done by recasting Einstein’s arguments for a 4-manifold structure a little. I will distinguish several different candidate definitions, in decreasing order of strength.

We say that a spacetime region is modelled by a classical manifold with classical metric to a desired accuracy if:

*Definition (a).* It is possible in principle to introduce test particles (i.e. particles not interfering with the measurement) whose trajectories can be measured to determine the geometry of the region (to the desired accuracy).

*Definition (b).* It is possible in principle to introduce particles whose trajectories can be

measured to determine the geometry of the region. The particles need not be test particles, but there must be an unambiguous mathematical means of recovering the geometry of the region (to the desired accuracy).

*Definition (c).* It is possible in principle to consider a sequence of similarly-prepared experiments in which particles are introduced, their trajectories are measured and these statistically determine the geometry of the region. The particles need not be test particles, but there must be an unambiguous mathematical means of recovering, from the data for the entire sequence of experiments, the geometry of the region in each experiment (to the desired accuracy).

*Remark.* In these definitions, the term ‘particle’ does not imply a classical point mass, but only an object with some degree of localizability. (One could contemplate definitions based not on observations of elements of the particles’ trajectories, but of other quantum observables. Such definitions, however, will not be considered here. They are less directly connected to the spacetime geometry, and so a proponent of such a definition would have to argue for an additional interpretational liberty.)

One may interpret (a) as saying that a classical spacetime has locally verifiable properties which are independent of the precise means of measurement. In (b), the classical spacetime exists, but attempts to measure its geometry, while successful, in general yield results which are dependent on the means of measurement: one must introduce some non-negligible amount of matter in order to effect the measurement, and so the geometry of the region does not have an operational existence independent of the means of measurement. In (c), a sort of statistical element may be present as well. (The motivation for (c) will appear presently.)

Now let us examine what happens in a negative energy regime, that is, when the accuracy of measurement is such that, on that length scale, negative energy densities are significant.

Condition (a) is the strongest and arguably the closest to the usual notion of a classical spacetime. Cases where it fails must be considered to at least raise serious questions about the application of classical general relativity. And this condition is directly forbidden by the operational energy conditions in a negative energy regime, since the energy density is related to the geometry by Einstein’s equation, and a measurement of the geometry accurate enough to reveal a negative energy density would require test particles and measuring apparatus of a greater positive energy. In essence, operational positivity precludes the existence of *test* particles in a negative energy-density regime, and this is why it is incompatible with (a).

Could spacetime be classical in sense (b) or (c) when negative energy densities are significant? In each of these cases, the probing particles must be about as massive as the source term whose geometrical effect they seek to reveal. Thus the problem of deducing the geometry is one of strong-field gravity. Recovering the spacetime from the data involves at least implicitly solving the joint evolution equations for the field, the measuring device and the spacetime. This would be difficult to pose properly and solve even if it were a classical problem. In the present context, though, the quantum nature of the detector, because of its finite mass and hence finite Compton length, as well as the quantum field observed, must be considered.

However, let us suppose a super-mathematician could overcome at least the mathematical difficulties involved. Still, there is good reason to think that spacetime will not be classical in the sense (b). This is because the state of the measuring device is subject to an uncertainty, manifested in the temporal uncertainty  $\Delta t$  in its resolving time. It is

uncertain precisely when the apparatus will turn on and turn off, and this contributes to an uncertainty in the measurement of the geometry. This sort of timing error was analysed at the end of section 4, where it was found that these dominate the measurement process unless one approaches the Planck scale. Of course, the analysis of section 4 concerns only one specific measuring device, and so there is the possibility that some other sort of device might be able to overcome this difficulty.

If one is willing to settle for a classical spacetime in the sense (c), then one can overcome the difficulty in the switching on and off of the device by statistical means, since the structure of the device and of the preparation of the experiment determines a probability distribution of on and off times. In other words, one could similarly prepare an ensemble of spacetimes, quantum fields and detectors, and examine the results of the experiments statistically. Even so, it is not clear that the resulting data can be well modelled by a classical spacetime or family of spacetimes. This is because, in the present context, the different possible states to which the state vector might reduce have differing geometries on a scale which is significant for the problem. (Even if the quantum field's state does not have this property, the state of the detector must, on account of its finite mass.) Thus one is confronted with the problem of understanding the dynamics of the reduction of the state vector. Whatever the solution of this is, there is no doubt of the quantum character of the process.

The arguments above cannot be considered as rigorously proving that spacetime cannot be classical in negative energy-density regimes. But they reveal a number of quantum measurement problems. How serious these problems are is a matter of judgment. Einstein created a theory of ineffable beauty in general relativity, and one should not seek to move beyond this without the most scrupulous consideration. But one should have the same reservation about abandoning operationalism.

These comments apply in particular to the 'semiclassical approximation', where the quantum field and the spacetime together satisfy the equation

$$G_{ab} = -8\pi G(T_{ab}^{\text{classical matter}} + \langle \hat{T}_{ab} \rangle). \quad (20)$$

The spacetime manifold and its metric are treated classically, and the quantity  $T_{ab}^{\text{eff}} = T_{ab}^{\text{classical matter}} + \langle \hat{T}_{ab} \rangle$  acts as an effective classical source. Kuo and Ford (1993) argued that negative energy-density states ought to be characterized by large fluctuations, and this made it doubtful that (20) could be an approximation to a deeper quantum theory of gravity and other fields. The present analysis uncovers another difficulty. Only between measurements of the energy density (or when the effective stress tensor satisfies the dominant energy condition), is it possible to regard (20) (together with the equation for the quantum field) as defining the evolution of the spacetime metric, including negative energy-density effects. In other words, *when negative energy-density effects are important, the semiclassical equation can be valid, as a local equation, only as long as there is no one there to check it!*

#### 6.4. An example: cosmic censorship

The foregoing discussion dealt with generalities about the structure of spacetime. I will close with a sketch of how these considerations might apply to a specific issue, cosmic censorship. As the cosmic censorship conjecture is not settled at the classical level, there is no pretense of my giving a complete treatment of the matter. However, even partial results are of interest, and the argument given here exhibits the physics of the operational weak energy condition briefly and clearly.

The cosmic censorship conjecture asserts that, under physically reasonable circumstances, spacetime singularities are not visible from great distances. A commonly

(but far from universally) held opinion on this is that it ought to be true for ‘classical’ matter fields (in generic circumstances), but is unlikely to hold when the negative energy densities engendered by quantum fields are taken into account.

Suppose that a singularity does arise in a region of negative energy densities due to quantum fields. We ask whether signals (say, photons) revealing this singularity might escape to great distances. Now, while there might well be *mathematical* null geodesics escaping from the singular region to infinity, the operational weak energy condition would prevent them from carrying *physical* photons energetic enough to *directly* reveal the geometry of the singular region. (For then the photons’ energy would have made the energy density in the singular region positive.) Thus a certain gross violation of the conjecture would be excluded.

Of course, this does not preclude the existence of other means of detecting the singularity, and does not touch on the ‘classical’ question of whether positive energy-density singularities might be visible from infinity, nor on whether negative energy densities in one region could give rise to a singularity in another, positive energy-density, region. The point is, whether cosmic censorship turns out to hold in general or not, the physics of the scenario discussed in the previous paragraph cannot be understood without considering quantum measurement issues.

In the terminology of the previous subsection, this example shows the inadequacy of a classical model in sense (a). One might not be bold enough to assert baldly that classical spacetime has broken down, but one must acknowledge the inadequacy of modelling the physics of the situation simply by the geometry of the null geodesics.

## 7. Conclusions

I have suggested that quantum fields obey a new sort of energy condition. The *operational weak energy condition* asserts that, locally, the energy of a field, together with the energy of an isolated device measuring or trapping the field’s energy, must be non-negative. A similar *operational dominant energy condition* asserts that any timelike component of the total energy density must be future-pointing.

There are good although at present not compelling arguments that at least linear quantum fields obey an operational weak energy condition, and perhaps an operational dominant energy condition. The gaps in the arguments come in part from technical difficulties in establishing Ford-type quantum inequalities in the most general circumstances, and partly from the order-of-magnitude nature of the inequalities of Salecker, Wigner *et al* on the masses of measuring devices. It also seems that the operational energy conditions could be violated if there were too many elementary field species.

The operational dominant energy condition immediately resolves two of the negative-energy pathologies listed in the introduction. It precludes the conversion of ordinary particles to tachyons. And at the level of Newtonian gravity, it forces gravitational fields to be attractive in the sense that  $\nabla \cdot \mathbf{g} \leq 0$ , for  $\mathbf{g}$  the gravitational acceleration field, since a measurement of  $\nabla \cdot \mathbf{g}$  is a measurement of the energy density. The remaining issues require more extensive discussion than can be given here. I only comment briefly that Grove (1988), in his resolution of the second-law problems raised by Ford (1978) and Davies (1982) in effect establishes a special case of the operational positivity of energy. I hope to discuss this, and the question of why perturbations do not cause quantum systems to decay into states with patches of negative energy together with positive-energy radiation, in a future publication.

My results cannot be considered good news for the semiclassical approximation for

gravity where there are negative energy densities. Even before this, Kuo and Ford (1993) had argued that such states would have large fluctuations. This means that the expectation values of the stress–energy do not reflect its eigenvalues, and on general grounds one would not expect a semiclassical approximation to be accurate. Still, one might have thought the approximation gives a rough classical response of the metric to the quantum field. But the present results restrict even this interpretation: where negative energy densities are significant, the semiclassical approximation can be valid only as long as no one is there to check it. While validity in this sense would be logically consistent, it is at odds with the hard-headed operational world-view which established the foundations of relativity and quantum mechanics.

The present results suggest that any attempt to understand the consequences of negative energy densities for gravity (such as Hawking evaporation, effect on singularity theorems, area theorem, positivity of Bondi and ADM energies) must take into account quantum measurement issues.

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