Unperformed experiments have no results

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This paper discusses correlations between the results of measurements performed on physical systems which are widely separated, but have interacted in the past. It is shown that quantum correlations are *stronger* than classical correlations. This property leads to the following paradox, known as Bell's theorem: Let us assume that the outcome of an experiment performed on one of the systems is independent of the choice of the experiment performed on the other. Now, let us try to imagine the results of alternative measurements, which could have been performed on the same systems *instead* of the actual measurements. Then there is no way of contriving these hypothetical results so that they will satisfy all the quantum correlations with the results of the actual measurements. However, the weaker classical correlations can be satisfied.

I. INTRODUCTION

Many experiments[1] have recently been performed to test predictions of quantum mechanics violating Bell's inequality.[2] The purpose of this paper is twofold. First, we give a simple proof of Bell's inequality, based on purely phenomenological arguments (no mention is made of "hidden variables" or similar superstitions). This inequality can be tested in experiments involving classical correlations or quantum correlations. It is shown that classical correlations respect Bell's inequality, but quantum correlations are *stronger* and lead to its experimental violation.

We then discuss the meaning of this result, again in purely phenomenological terms. The discussion involves a comparison of the results of experiments which were actually performed, with those of hypothetical experiments which could have been performed but were not. It is shown that it is *impossible to imagine* the latter results in a way compatible with (a) the results of the actually performed experiments, (b) long range separability of results of individual measurements, and (c) quantum mechanics.

In the Appendix, we show that a Bell inequality can be constructed (and is violated by quantum mechanics) for any pair of states of any two dynamical variables. However, for the sake of conciseness, the main part of this paper will deal with the simple case of a pair of correlated spins or, classically, of correlated macroscopic angular momenta.

II. CORRELATED DYNAMICAL VARIABLES

Consider a bomb, initially at rest, which explodes into two asymmetric parts carrying angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2 = -\mathbf{J}_1$ (see Fig. 1). An observer detects the first fragment and measures

$$r_\beta = \operatorname{sign}(\beta \cdot \mathbf{J}_2),$$

where α is a unit vector in an arbitrary direction, chosen by that observer. Likewise, a second observer detects the other fragment and measures

$$r_\beta \operatorname{sign}(\beta \cdot \mathbf{J}_2),$$

where β is another unit vector, chosen by the second observer.

Now assume that the directions of $\mathbf{J}_1$ and $\mathbf{J}_2$ are unpredictable and randomly distributed. Then, if the experiment is repeated N times, the averages

$$\langle r_\alpha \rangle = (1/N) \sum_j r_{j\alpha}$$

and

$$\langle r_\beta \rangle = (1/N) \sum_j r_{j\beta}$$

are both close to zero (typically, they are of the order of $1/\sqrt{n}$). Here, $r_{j\alpha}$ and $r_{j\beta}$ ($j = 1, \ldots, N$) denote r_α and r_β of the jth bomb. We shall henceforth omit the subscript j for brevity.

On the other hand, if the observers compare their results after the latter have been obtained, the average

$$\langle r_\alpha r_\beta \rangle = (1/N) \sum r_\alpha r_\beta$$

may differ from zero. For instance, if $\alpha = \beta$, we shall always have $r_\alpha = -r_\beta$ so that $\langle r_\alpha r_\beta \rangle = -1$. Quite generally, if we consider a unit sphere cut by an equatorial plane perpendicular to α, we shall have $r_\alpha = 1$ if $\mathbf{J}_1$ points through one of the hemispheres and $r_\alpha = -1$ if it points through the other. Likewise a second equatorial plane, perpendicular to β determines the regions with $r_\beta = \pm 1$. The unit sphere is thereby divided by these two equatorial planes into four parts, with $r_\alpha r_\beta = \pm 1$. Their areas are in the ratio $\theta : \pi - \theta$, where θ is the angle between α and β. If $\mathbf{J}_1$ is randomly distributed, we obtain, for large N,

$$\langle r_\alpha r_\beta \rangle = [\theta - (\pi - \theta)]/\pi = -1 + 2\,\theta/\pi.$$

Next, consider two spin-1/2 particles in a singlet state, flying apart from each other in a way that spin is conserved. Our observers now measure $r_\alpha = 2\alpha \cdot \mathbf{s}_1$ and $r_\beta = 2\beta \cdot \mathbf{s}_2$. As in the previous case, r_α and r_β are ± 1, and $\langle r_\alpha \rangle$ and $\langle r_\beta \rangle$ tend to zero for large N, but $\langle r_\alpha r_\beta \rangle$ will not in general be zero: quantum mechanics tells us that, for large N, we should find

$$\langle r_\alpha r_\beta \rangle = \langle \psi | 2\alpha \cdot \mathbf{s}_1 \quad 2\beta \cdot \mathbf{s}_2 | \Psi \rangle.$$

In the singlet state, $\mathbf{s}_2 \, |\Psi\rangle = -\mathbf{s}_1 |\Psi|$, and as

$$[(-2\alpha \cdot \mathbf{s}_1)(2\beta \cdot \mathbf{s}_1) = (-\alpha \cdot \sigma)(\beta \cdot \sigma) = -\alpha \cdot \beta,$$

we obtain

$$\langle r_\alpha r_\beta \rangle = -\alpha \cdot \beta = -\cos\theta.$$

Figure 2 shows the classical and the quantum correlations. We shall denote both of them as $C(\alpha,\beta)$. Note that

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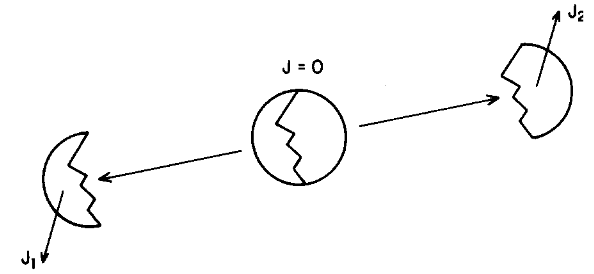

Fig. 1. A bomb, initially at rest, explodes into two fragments carrying opposite angular momenta.

$$|C(\alpha,\beta)_{\text{quantum}}| \geqslant |C(\alpha,\beta)_{\text{classical}}|.$$

This fact has far reaching consequences.

III. BELL'S INEQUALITY

If the observers A and B are far away from each other, we expect intuitively that the results obtained by A are independent of what B is doing and vice versa. For instance, if B had oriented his apparatus along some other direction β', the results obtained by A would have been the same: Not only $\langle r_\alpha \rangle$ would have remained zero, but *each individual* $r_{j\alpha}(j = 1, \cdots, N)$ would have remained unchanged. Indeed, the bomb fragments or the spin-1/2 particles may have started their flight *before* A and B orient their apparatuses along α and β. It thus seems reasonable to assume that B's choice of β cannot affect the result obtained by A, and vice versa.

Let us now try to imagine[3] what could have been the results of measurements by A and B along other directions α' and β'. It is of course impossible to know for sure these results (unless $\alpha' = \pm\alpha$ or $\pm\beta$, etc.) because the experiments were not performed. However, they *could* have been performed (instead of the measurements along α and/or β) and the results could have been only $r_{\alpha'} = \pm 1$ and $r_{\beta'} = \pm 1$.

We can therefore construct a table including both the actual and the hypothetical results (see Table I). We do not know how to fill the last two rows in this table. However, there are only 4^N different ways of doing this, that is, there are only 4^N different tables which can be imagined. We shall soon see that *none* of them may be acceptable.

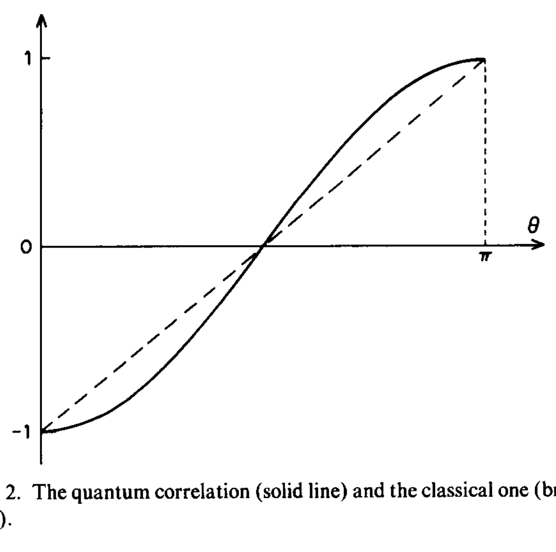

Fig. 2. The quantum correlation (solid line) and the classical one (broken line).

Table I. Comparison of actual and hypothetical measurements.

	Exp. No.	1	2	3	·	·	·	N
Experiments actually performed	$r_\alpha =$	+	+	−	·	·	·	·
	$r_\beta =$	−	+	−	·	·	·	·
Alternative unperformed experiments	$r_{\alpha'} =$	?	?	?	·	·	·	·
	$r_{\beta'} =$	?	?	?	·	·	·	·

To be acceptable, a table must have (for large enough N) $\langle r_{\alpha'} \rangle = \langle r_{\beta'} \rangle = 0$ and moreover

$$\langle r_\alpha r_{\beta'} \rangle = C(\alpha,\beta'),$$
$$\langle r_{\alpha'} r_\beta \rangle = C(\alpha',\beta),$$
$$\langle r_{\alpha'} r_{\beta'} \rangle = C(\alpha',\beta'),$$

because the unperformed experiments *could* have been performed, and then should have given results consistent with the known laws of physics.

Now it is not difficult to see that

$$r_\alpha r_\beta + r_\alpha r_{\beta'} + r_{\alpha'} r_\beta - r_{\alpha'} r_{\beta'} \equiv \pm 2,$$

for any choice of $r_\alpha, \cdots, r_{\beta'}$ (all ± 1). Therefore

$$(1/N)|\textstyle\sum(r_\alpha r_\beta + r_\alpha r_{\beta'} + r_{\alpha'} r_\beta - r_{\alpha'} r_{\beta'})| \leqslant 2.$$

But the left-hand side of this expression is just $|\langle r_\alpha r_\beta \rangle + \langle r_\alpha r_{\beta'} \rangle + \langle r_{\alpha'} r_\beta \rangle - \langle r_{\alpha'} r_{\beta'} \rangle|$. It follows that

$$|C(\alpha,\beta) + C(\alpha,\beta') + C(\alpha',\beta) - C(\alpha',\beta')| \leqslant 2.$$

This is Bell's inequality.

It is easy to show that the latter is violated by the quantum correlation function $C(\alpha,\beta) = -\alpha\cdot\beta$. The maximum violation occurs when all the directions are coplanar, with the (α',β') angle equal to 135° and the three others to 45°. Then

$$\alpha' \cdot \beta = \beta \cdot \alpha = \alpha \cdot \beta' = -\alpha' \cdot \beta' = 1/\sqrt{2},$$

and the left-hand side of Bell's inequality is $2\sqrt{2}$.

Actually, it is more instructive[4] to take $\alpha = \beta$, with α' and β' at an angle θ on each side, so that $\alpha\cdot\beta' = \alpha'\cdot\beta = \cos\theta$ and $\alpha'\cdot\beta' = \cos 2\theta$. The left-hand side of Bell's inequality becomes

$$|-1 - 2\cos\theta + \cos 2\theta| = |-2 - 2\cos\theta(1 - \cos\theta)|$$

which is larger than 2 for any $\theta < 90°$.

On the other hand, if we take the *classical* correlation function $-1 + 2\theta/\pi$, instead of $-\cos\theta$, we get

$$-1 - 2(1 - 2\theta/\pi) + (1 - 4\theta/\pi) = -2$$

exactly. The difference between these results is due to the fact that for small θ, the classical correlation function *falls faster* than the quantum-mechanical one.

IV. DISCUSSION AND OUTLOOK

There are two possible attitudes in the face of these results. One is to say that it is illegitimate to speculate about unperformed experiments. In brief "Thou shalt not think." Physics is then free from many epistemological difficulties. For instance, it is not possible to formulate the EPR paradox.[5]

Alternatively, for those who cannot refrain from thinking, we can abandon the assumption that the results of measurements by A are independent of what is being done by B. In other words, we should not have written $r_{j\alpha}$, but $r_{j\alpha(\beta)}$ or rather $r_{j\alpha(\beta\gamma\delta\cdots)}$ where the additional subscripts $\gamma,\delta,\cdots$ refer to possible experiments on anything else in the world which has been correlated with the history of the jth particle.

This conclusion is surprising. Physicists are used to thinking in terms of isolated systems whose behavior is independent of what happens in the rest of the world (contrary to social scientists, who cannot isolate the subject of their study from its environment). Bell's theorem tells us that such a separation is impossible for individual experiments, although it still holds for averages:

$$(1/N)\sum_j r_{j\alpha(\beta\gamma\delta\cdots)} = 0$$

and

$$(1/N)\sum_j r_{j\alpha(\beta\gamma\delta\cdots)} r_{j\beta(\alpha\gamma\delta\cdots)} = -\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$$

independently of γ,δ, etc.

As it often happens, the subtlety of nature beggars the human imagination.

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APPENDIX

The reader may have the impression that Bell's theorem is limited to spin correlations (or photon correlations[6] which have the same algebra). In this Appendix, we show that it can easily be generalized to any pair of states of any two suitably correlated systems.

Let $|m\rangle$ and $|n\rangle$ be two orthogonal states of a physical system. Consider the family of operators

$$F(\alpha) = e^{i\alpha}|m\rangle\langle n| + e^{-i\alpha}|n\rangle\langle m|.$$

These operators have eigenvalues ± 1 corresponding to eigenvectors $(e^{i\alpha/2}|m\rangle \pm e^{-i\alpha/2}|n\rangle)/\sqrt{2}$, and 0 corresponding to all states orthogonal to $|m\rangle$ and $|n\rangle$. Thus any measurement of $F(\alpha)$ shall yield the result $r_\alpha = \pm 1$ or 0. Note that

$$[F(\alpha), F(\alpha')] = 2i\sin(\alpha-\alpha')(|m\rangle\langle m| - |n\rangle\langle n|)$$

does not vanish (unless $\alpha - \alpha'$ is a multiple of π) so that it is impossible to measure both operators simultaneously.

Likewise, let $|\mu\rangle$ and $|\nu\rangle$ be two orthogonal states of another physical system and let us consider the family of operators

$$\Phi(\beta) = e^{i\beta}|\mu\rangle\langle\nu| + e^{-i\beta}|\nu\rangle\langle\mu|.$$

Any measurement of $\Phi(\beta)$ shall yield the result $r_\beta = \pm 1$ or 0. Now, let

$$|\Psi\rangle = (1/\sqrt{2})\,(|m\rangle|\nu\rangle - |n\rangle|\mu\rangle)$$

be the wave function describing an ensemble of pairs of these systems. The elements of each pair can be well separated from each other, yet they are correlated. A straightforward calculation gives

$$\langle\Psi|F(\alpha)\Phi(\beta)|\Psi\rangle = -\cos(\alpha-\beta),$$

and this must be the limiting value of $N^{-1}\sum(r_\alpha r_\beta)$ for large N.

On the other hand, if we consider the possibility of measuring $F(\alpha')$ and/or $\Phi(\beta')$ instead of $F(\alpha)$ and/or $\Phi(\beta)$ we get, as before,

$$|r_\alpha r_\beta + r_\alpha r_{\beta'} + r_{\alpha'} r_\beta - r_{\alpha'} r_{\beta'}| \leqslant 2.$$

Indeed, if none of $r_\alpha, \cdots, r_{\beta'}$ is zero, this is exactly 2. If one or more is zero, the result can also be 1 or 0. So, we again obtain Bell's inequality, which is violated by the above cosine correlation.

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