An elementary solution of the Maxwell equations for a time-dependent source

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Abstract

We present an elementary solution of the Maxwell equations for a time-dependent source consisting of an infinite solenoid with a current density that increases linearly with time. The geometrical symmetries and the time dependence of the current density make possible a mathematical treatment that does not involve the usual technical difficulties, thus making this presentation suitable for students that are taking a first course in electromagnetism. We also show that the electric field generated by the solenoid can be used to construct an exact solution of the relativistic equation of motion of the electron that takes into account the effect of the radiation. In particular, we derive, in an almost trivial way, the formula for the radiation rate of an electron in circular motion.

1. Introduction

In a basic electromagnetism course the generation of static electric and magnetic fields is illustrated by means of idealized sources, which have a simple geometrical shape. Although these idealized sources are usually of infinite extent, thus making their experimental implementation impossible, they introduce symmetries that make it possible to obtain the electric or magnetic fields without the necessity of elaborate calculations. This happens, for instance, in the case of the electrostatic field generated by a capacitor of infinite parallel plates and in the case of the homogeneous magnetic field generated by a flat unbounded sheet with a uniform time-independent current density.

A basic course treating electromagnetism usually ends with Maxwell’s equations, but in contradistinction to the case of a time-independent source, the solution of Maxwell’s equations for a time-dependent source turns out to be too elaborate. Even though the general solution can be expressed by means of an explicit formula, the latter has a rather complicated structure, mainly because of the retardation condition contained in it. In other words, although the physical meaning of the retarded effect can be easily understood by students in a qualitative way, the mathematical treatment of the retardation condition introduces, even with idealized sources of simple geometry [1], some rather elaborate mathematical concepts.

In this paper we consider a detailed study of an exact solution of Maxwell’s equations for a relatively simple time-dependent source in which it is not necessary to deal explicitly
with retarded effects. The source consists of an infinitely long solenoid fed with a current density that increases linearly with time. The solution is of pedagogical interest because the geometrical symmetries, together with the special time dependence of the current, allow a mathematical treatment that does not involve technical difficulties for students studying a first course in electromagnetism.

Another difficult aspect of teaching in an elementary way, related to time-dependent sources, is the radiation emitted by an accelerated electron. Here we propose to use the electric field generated by the solenoid as a way of obtaining insight regarding the effect of the radiation on the electron movement. Since the radiation transports energy and momentum, the students can easily understand, in a qualitative way, that the emission of radiation affects the motion of the electron. However, the study of the electron fields and the derivation of the equation of motion are too elaborate for students taking a basic course treating electromagnetism or electrodynamics. For this reason, here we propose to start from the equation of motion to learn about the radiation emitted by the electron in the special case of a mono-energetic electron in a circular orbit. As it is known, this movement is also of great practical importance, since it is related to particle accelerators and synchrotron radiation.

If the emission of radiation is neglected, the electron describes a circle of a fixed radius in a homogeneous magnetic field; however, because of the radiation, the electron trajectory is an inward spiral. As it will be shown here, the electric field of the infinitely long solenoid can exactly compensate the effect of the radiation on the electron movement, in such a way that the electron remains in a circle of fixed radius. This in turn allows us to derive, in an almost trivial way, the formula for the rate of radiation emitted by the electron in this particular but important case.

This paper is organized as follows.

Section 2 deals with the static magnetic field generated by the infinitely long solenoid fed with a time-independent current. There are two reasons that justify starting with the static case. The first one is that even though in this case the magnetic field can be found everywhere, normally in textbooks the magnetic field is calculated only in the axis of the solenoid. The second reason is of a heuristic nature, because the static case allows us to guess the fields for a current that increases linearly with time.

In section 3 we start by conjecturing the analytic form of the electric and magnetic fields generated by the solenoid fed with a current that increases linearly with time. Then, by expressing the fields in terms of its Cartesian components, it is shown that they satisfy Maxwell’s equations inside and outside the solenoid. Now, since the fields also satisfy the boundary conditions across the surface of the solenoid, they are indeed the physical solution of the Maxwell equations for such a source.

Section 4 deals with the fields expressed in cylindrical coordinates, which are advantageous because they allow one to verify Maxwell’s equations everywhere, including on the surface of the solenoid. Moreover, since it is instructive from a pedagogical point of view, in this section we also discuss the energy conservation law for these specific forms of the electric and magnetic fields associated with the infinitely long solenoid.

Finally, in section 5, we present the equation of motion of the electron when the effect of the radiation is taken into account, the Lorentz–Dirac equation, and we apply it to the movement of the electron in the superposition of the field outside the solenoid and an external homogeneous magnetic field. As it is shown here, for these external fields the equation of motion admits an exact solution that represents the motion of a monoenergetic electron in a circular orbit. The movement takes place in a circle centred at the solenoid’s axis and contained in a plane orthogonal to it. The electron can have any velocity, including velocities as close to the velocity of light as we like, by properly choosing the current in the solenoid.

2. The magnetostatic field

Figure 1 shows a section of the infinitely long solenoid. The solenoid is formed by a cylindrical conducting sheet of radius \( b \), where a time-independent current flows around it. The direction
Figure 1. A section of the infinitely long solenoid of radius $b$. The solenoid carries a density of current that goes into the plane of the paper in the upper part of the solenoid. We show a circular path $C$ and the three components of the magnetic field at a point $P$ on it; a closed cylindrical surface and the exterior unit normals at points $P_1$ and $P_2$ on its caps; and the rectangular paths $a-b-c-d-e-f$ and $a-b-c-f$.

of the current flow is perpendicular to the plane of the paper, and for definiteness it will be assumed to flow into the paper in the upper part and out of the paper in the lower part of the solenoid.

In textbooks the magnetic field $B$ is calculated in the axis of the solenoid by means of the Biot–Savart law, and it turns out to be given by

$$B = \mu_0 J \hat{z}$$  \hspace{1cm} (1)

where $J$ is the magnitude of the surface current density per unit length along the solenoid, $\mu_0$ is the permeability of free space and $\hat{z}$ is the unit vector that points to the left along the solenoid’s axis as shown in figure 1. However, the magnetic field $B$ can be determined everywhere, not only in the axis of the solenoid, by using the geometrical symmetries of the problem. This approach has the further advantage of making it possible to find the magnetic field $B$ outside the solenoid in an exact way and without any further assumptions, a point that is not usually handled correctly in textbooks\(^1\). In what follows, we give a simple derivation of $B$ everywhere, based in the integral form of the Maxwell equations:

$$\oint B \cdot dl = \mu_0 I_{enc}$$  \hspace{1cm} (2)

$$\oint B \cdot \hat{n} \, dS = 0$$  \hspace{1cm} (3)

where the line integral in equation (2) is carried out on any closed path and $I_{enc}$ is the current enclosed by the path; whereas the surface integral in equation (3) is carried out on any closed surface $S$ with exterior unit normal $\hat{n}$.

The infinitely long solenoid presented in figure 1 has two geometrical symmetries, namely circular, around the axis of the solenoid, and translational, along any straight line parallel to the solenoid’s axis. In figure 1 we present a circular path $C$ of radius $r$ centred at the solenoid’s axis and contained in a plane orthogonal to it. In general, the magnetic field $B$ has three components at any point, such as $P$, on the circular path: a radial component $B_\rho$, a tangential component $B_\phi$, and a $z$-component $B_z$. Because of the symmetry around the solenoid’s axis,

\(^1\) In fact, most textbooks need to use plausibility arguments in order to find the magnetic field outside the solenoid. For example, Purcell [2], after showing that $B$ has a constant magnitude outside the solenoid, argues that the solenoid could be made as slender as one pleases, and that it would be strange if a solenoid of vanishing diameter could still create a field that fills all space. Other textbooks simply make incorrect statements. This even happens in excellent textbooks, such as Griffiths [3], where it is argued that the constant value of $B$ outside the solenoid is zero, since $B$ must go to zero far away from the solenoid. Clearly, this argument is not applicable to an infinite source, and the electric field of an infinite plane that carries a constant surface charge is a clear counter-example.
the magnitude of each of these components is constant in the circular path. Now, applying equation (2) to the circular path \( C \) shown in figure 1, it follows that

\[
(2\pi r) B_\rho = 0.
\]

Therefore \( B \) can only have components \( B_\rho \) and \( B_z \) which differ from zero.

The component \( B_\rho \) also vanishes because of the symmetries. Let us apply equation (3) to the closed cylindrical surface of radius \( r \) shown in figure 1. Since \( B \) is the same at all points of the straight line \( L \), and since the exterior normal at \( P_1 \) points in the opposite direction with respect to the exterior normal at \( P_2 \), the contributions of the points \( P_1 \) and \( P_2 \) in the integral over the discs cancel each other exactly. Therefore, the integral over the closed cylindrical surface presented in figure 1 reduces to the integral over the cylindrical part only. Then, equation (3) becomes

\[
(2\pi r L) B_\rho = 0
\]

where \( L \) is the length of the cylindrical surface. The magnetic field is, therefore, necessarily parallel or anti-parallel to the solenoid’s axis. This discussion is, of course, valid inside as well as outside the solenoid.

In order to determine the magnitude \( B \) of the magnetic field, it is useful to apply equation (2) to the rectangular paths that appear in figure 1. Let us consider first the oriented rectangular path \( a \rightarrow b \rightarrow c \rightarrow f \rightarrow a \) contained completely inside the solenoid. If we denote by \( B \) the magnitude of the magnetic field in the side \( a \rightarrow b \) and by \( B_1 \) the corresponding magnitude in the side \( c \rightarrow f \), then equation (2) gives

\[
-B L + B_1 L = 0
\]

where \( L \) is the length of the side \( a \rightarrow b \). In other words, the magnetic field \( B \) is homogeneous inside the solenoid, and it is given there by (1).

If equation (2) is now applied to the oriented path \( a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow a \) and the magnitude of \( B \) in the side \( e \rightarrow d \) is denoted by \( B_2 \), we obtain

\[
-B L + B_2 L = -\mu_0 J L.
\]

The minus sign on the right-hand side of (7) is due to the use of the right-hand rule, since the current in the upper part of the solenoid flows into the plane presented in figure 1, that is, opposite to the normal of the area enclosed by the path \( a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow a \). From equations (7) and (1) it then follows that the magnetic field \( B \) vanishes identically outside the infinitely long solenoid.

3. A linearly time-increasing current

Let us now discuss the fields generated by the solenoid presented in figure 1 when it is fed with a current that increases linearly with time \( t \). This current configuration can be generated, for instance, by attaching a uniform surface charge density over a non-conducting cylinder, which is then rotated with constant angular acceleration by means of a mechanical device. Such a mechanism can, of course, be sustained only for a finite time interval. However, the idealization of that time interval being arbitrarily large introduces great simplifications in the mathematical treatment of the problem, making, in fact, possible a closed exact solution of the Maxwell equations for a non-trivial time-dependent current distribution.

Our approach to the problem is to guess the values of the electric and magnetic fields from geometrical and physical grounds (instead of calculating them starting from the general solution of the Maxwell equations) and then verify that they are in fact the fields generated by the solenoid.

Since the charge density is everywhere zero, the only source of the fields is the time-dependent current of the solenoid. Let us assume that, like in the case of a static current, the
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Figure 2. Circumference $C_\rho$ of radius $\rho$ larger than the solenoid’s radius $b$, to which Faraday’s law is applied. The electric field points in the clockwise direction, while the surface current in the solenoid flows in the counterclockwise direction.

magnetic field is confined inside the solenoid, is homogeneous, points along the solenoid’s axis in the same direction as the unit vector $\hat{z}$, and is given by (1), where

$$J = \alpha t$$

and $\alpha$ is a positive number. Because of the Maxwell equations, a time-dependent magnetic field requires the existence of an electric field. Now, the geometrical symmetries suggest that the electric field is tangent to the circumferences centred at the solenoid’s axis and contained in a plane orthogonal to it. Moreover, since the points of a given circumference are indistinguishable between them, the electric field $\mathbf{E}$ must have a fixed magnitude $E$ over the circumference. The determination of $E$ is carried out by applying Faraday’s law of induction to path $C_\rho$ shown in figure 2, where $C_\rho$ is a circumference of radius $\rho$ larger than the solenoid’s radius $b$.

In figure 2 the magnetic field points in the same direction as the unit normal $\hat{n}$ (that coincides with the unit vector $\hat{z}$) of the area enclosed by the circumference $C_\rho$, that is, towards the reader. When Faraday’s law

$$\int_{C_\rho} \mathbf{E} \cdot d\ell = -\frac{d}{dt} \int_S \mathbf{B} \cdot \hat{n} \, dS$$

is applied to the circle $C_\rho$ presented in figure 2 in a counterclockwise direction, it gives

$$2\pi \rho E = \frac{d}{dt} (\pi b^2 B).$$

However, since by assumption $B$ is given by $B = \mu_0 \alpha t$, it follows that

$$E = \frac{\mu_0 \alpha b^2}{2} \frac{1}{\rho} \quad \text{for} \quad \rho > b.$$ (11)

This electric field is time independent and therefore it cannot give rise to a magnetic field, which is consistent with the assumption that there is no magnetic field outside the solenoid.

If Faraday’s law (9) is applied to a circle of radius $\rho$ less than the radius of the solenoid the magnitude of the electric field turns out to be

$$E = (\mu_0 \alpha/2) \rho$$ (12)

which is also time independent. Introducing a Cartesian coordinate system where the $z$-axis coincides with the solenoid’s axis, the electric and magnetic fields have the following
components:

\[
E_x = \left(\mu_0 \alpha b^2 / 2\right) \frac{y}{x^2 + y^2} \\
E_y = -\left(\mu_0 \alpha b^2 / 2\right) \frac{x}{x^2 + y^2} \\
E_z = 0 \\
B_x = B_y = B_z = 0
\]  
(13)

outside the solenoid, and

\[
E_x = \left(\mu_0 \alpha / 2\right) y \\
E_y = -\left(\mu_0 \alpha / 2\right) x \\
E_z = 0 \\
B_x = B_y = 0; \quad B_z = \mu_0 \alpha t
\]  
(14)

inside it.

Maxwell’s equations, in the case of a vanishing density of charge, are as follows:

\[
\nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}.
\]

(15)

From expressions (13) and (14) it is immediate to check that these equations are satisfied inside as well as outside the solenoid. Therefore, in order to show that the fields (13) and (14) are indeed the fields generated by the solenoid with a current that increases linearly with time, it is enough to verify that they satisfy the boundary conditions at the solenoid’s surface.

As it is known, the boundary conditions associated with the equations \(\nabla \cdot \mathbf{E} = 0\) and \(\nabla \cdot \mathbf{B} = 0\) mean that the normal components of \(\mathbf{E}\) and \(\mathbf{B}\) must be continuous across the surface of the solenoid. These conditions are certainly satisfied because the normal components of \(\mathbf{E}\) and \(\mathbf{B}\) vanish identically at the solenoid’s surface, since \(\mathbf{E}\) is tangent to the circles presented in figure 2, and \(\mathbf{B}\) is parallel to the solenoid’s axis inside the solenoid and vanishes identically outside it.

The boundary condition associated with the Maxwell equation for \(\nabla \times \mathbf{E}\) is as follows:

\[
\hat{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0
\]

(16)

where \(\hat{n}\) is the normal that points outwards from the solenoid’s surface, \(\mathbf{E}_2\) is the electric field immediately outside the solenoid and \(\mathbf{E}_1\) is the electric field immediately inside the solenoid. Equation (16) is satisfied because from (13) and (14) it follows that the electric field \(\mathbf{E}\) is continuous across the surface of the solenoid \(x^2 + y^2 = b^2\), that is, \(\mathbf{E}_2 = \mathbf{E}_1\).

The boundary condition associated with equation (15) for \(\nabla \times \mathbf{B}\) is in this case

\[
\hat{n} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0 J \hat{\varphi}
\]

(17)

where the symbols in the left-hand side have the same meaning as in equation (16), and \(\hat{\varphi}\) is the unit vector that determines the direction of the surface density of current \(J\) at the solenoid’s surface; \(\hat{\varphi}\) is tangent to the circle of the solenoid presented in figure 2 and points in the counterclockwise direction. Equation (17) is immediate since \(\mathbf{B}_2 = 0\) and \(\mathbf{B}_1\) is given by (1).

As a final comment in this section, let us point out that the term ‘electrostatic field’ should not be used in connection with the electric field (11), since in addition to being time independent, an electrostatic field must satisfy the condition of not having closed force lines. This point is not usually stressed in textbooks, and may lead to confusion for students that make the above identification, since they can easily verify from (13) that \(\nabla \times \mathbf{E} = 0\) and that, however, the line integral of the electric field on any closed circular path of radius \(\rho > b\) is not zero but a constant. The point, of course, is that an irrotational field must have \(\nabla \times \mathbf{E} = 0\) everywhere, which is not the case inside the solenoid.
4. Cylindrical coordinates

Due to the geometry of the solenoid, the cylindrical coordinates \((\rho, \phi, z)\) with the \(z\)-axis coinciding with the solenoid’s axis appear as the most suitable coordinate system to describe the electric and magnetic fields generated by the infinitely long solenoid. In this case both fields have only one component. The electric field \(E\) points along \(-\hat{\phi}\), where \(\hat{\phi}\) is the unit vector associated with the coordinate \(\phi\); whereas the magnetic field \(B\) points along \(\hat{z}\), the unit vector associated with the \(z\) coordinate. These fields can be written in a compact way by means of the step function \(\theta(x)\), which takes the value zero for \(x < 0\) and the value 1 for \(x > 0\). From equations (13) and (14) it follows that in cylindrical coordinates the electric and magnetic fields are given by

\[
E = -\frac{\mu_0 \alpha}{2} \left( \rho - \theta(\rho - b) \left( \rho - \frac{b^2}{\rho} \right) \right) \hat{\phi},
\]

\[
B = \mu_0 \alpha t \left[ 1 - \theta(\rho - b) \right] \hat{z}.
\]

(18)

In order to facilitate the verification of Maxwell’s equations (15), here we give the expressions for \(\nabla \cdot E\) and \(\nabla \times E\) in cylindrical coordinates [3], namely

\[
\nabla \cdot E = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\rho) + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z},
\]

\[
\nabla \times E = \hat{\rho} \left[ \frac{1}{\rho} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} \right] + \hat{\phi} \left[ \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} \right] + \hat{z} \left[ \frac{\rho}{\rho} \frac{\partial (\rho E_\phi)}{\partial \rho} - \frac{\partial E_\phi}{\partial \phi} \right].
\]

(19)

The derivative of the step function \(\theta(\rho - b)\) that appears in (18) gives rise to a Dirac delta function \(\delta(\rho - b)\) according to

\[
d \theta(\rho - b) = \delta(\rho - b) \frac{d \rho}{d \rho}.
\]

(20)

The Dirac delta function is useful for writing the idealized current density \(J\) of the solenoid, since it takes the form

\[
J = \alpha t \delta(\rho - b) \hat{\phi}.
\]

(21)

The verification of the Maxwell equations (15) for the fields (18) and the current density (21) is straightforward. In the calculation of \(\nabla \times E\) appears a term proportional to

\[(\rho^2 - b^2) \delta(\rho - b)\]

(22)

which is equal to zero, since the factor \((\rho^2 - b^2)\) in front of the delta function vanishes for \(\rho = b\).

Due to the simple form of the electric and magnetic fields given in (18), it is instructive to analyse the law of energy conservation inside the solenoid, which in this case reads

\[
\frac{\partial u}{\partial t} + \nabla \cdot S = 0
\]

(23)

where the density of energy \(u\) is given by

\[
u = \frac{1}{2} \varepsilon_0 E^2 + \frac{B^2}{2 \mu_0}
\]

(24)

and the Poynting vector is given by

\[
S = \frac{1}{\mu_0} (E \times B).
\]

(25)

Let us integrate equation (23) over the volume \(V\) of a cylinder of height \(h\) and radius \(\rho < b\) concentric with the solenoid. Using the Gauss–Ostrogodski (or divergence) theorem, equation (23) becomes

\[
\frac{dU}{dt} = -\int_V \nabla \cdot S \, d^3 x = -\int_S S \cdot \hat{n} \, d \Sigma
\]

(26)
where $U$ is the total electromagnetic energy contained in the volume $V$ of the cylinder and $\hat{n}$ is the outward unit normal to the surface $\Sigma$ of the cylinder. Now, from (18) it follows that inside the solenoid the electric and magnetic fields are given by

$$ E = -\frac{\mu_0 \alpha \rho}{2} \hat{\varphi} \quad B = \mu_0 \alpha t \hat{z}. $$

(27)

Although the electric field contributes to the electromagnetic energy $U$, its contribution is time independent, and therefore it is irrelevant for the derivative $dU/dt$ that appears in (26). On the other hand, the magnetic field is homogeneous inside the cylinder. Therefore, from (24) and (27) it follows that

$$ \frac{dU}{dt} = \mu_0 \alpha^2 V t. $$

(28)

This equation states that the total energy $U$ increases linearly with time. Now, let us examine the electromagnetic energy flux across the surface $\Sigma$ of the cylinder, given by the integral over $\Sigma$ that appears in equation (26). From (27) and (25) it follows that the Poynting vector $S$ is given by

$$ S = -\mu_0 \frac{\alpha^2}{2} \rho t \hat{\rho}. $$

(29)

However, the outward normal $\hat{n}$ at the end disc caps of the cylinder is orthogonal to $S$, and therefore they do not contribute to the integral over the surface $\Sigma$ in equation (26). Thus, only the cylindrical surface with outward unit normal $\hat{\rho}$ contributes, and because of (26) its contribution turns out to be

$$ -(2\pi \rho L) \left( -\mu_0 \frac{\alpha^2}{2} \rho t \right) = \mu_0 \alpha^2 V t. $$

(30)

In other words, the increase per unit of time of the electromagnetic energy contained in the cylinder is precisely the same as the energy flux into the cylinder through its surface. Note also that $S = 0$ outside the solenoid (since the magnetic field $B$ produced by the surface current (8) vanishes identically there), and therefore there is no energy escaping away to infinity in the form of radiation. This result has been shown explicitly in [1], where it is shown to be a consequence of the specific time dependence in equation (8). Finally, note that there is no field energy escaping to infinity as a result of the interference between the electric field of the solenoid and the magnetic field of the charge. This can be seen using a closed surface that consists of a cylindrical surface of a very large radius $\rho$ concentric with the solenoid and two discs located at $z = \pm \infty$. If the flux of the Poynting vector over a ribbon of width $d\zeta$ on the cylindrical surface is considered, the linear $\rho$ dependence of the surface element $dA = 2\pi \rho d\zeta$ cancels out with the $\rho^{-1}$ dependence of the electric field (11), and therefore the flux over the ribbon is essentially proportional to the value of the magnetic field of the charge, which tends to zero when the radius $\rho$ goes to infinity. A similar analysis holds for the flux over the discs at $z = \pm \infty$.

5. A monoenergetic electron in a circular orbit

In this section we will use the time-independent electric field generated outside the solenoid to construct an exact solution of the relativistic equation of motion of the electron, corresponding to a monoenergetic electron in a circular orbit. The equation of motion that fully takes into account the effect of the radiation in the movement of the electron is given by

$$ \dot{v}^\mu = (e/m)V^\mu v_\mu + \tau_0 (\dot{v}^\mu - \dot{v}^\nu \dot{v}_\nu /c^2), $$

(31)

where $e < 0$ represents the electric charge of the electron, $m$ denotes its mass, $c$ is the velocity of light and $\tau_0$ is given by

$$ \tau_0 = \frac{e^2}{6\pi \varepsilon_0 mc^3}. $$

(32)
This equation, usually called the Lorentz–Dirac equation, is derived in advanced books dealing with electrodynamics [4]. However, for the purpose of this paper it is not necessary that the students master the formalism of the relativity theory, since it is enough that they understand the meaning of the different symbols that appear in (31). In our opinion the study of the present solution of (31) is stimulating for students even if they have little knowledge about the theory of relativity. In particular, this problem is especially motivating for those students that have an inquisitive spirit and want to know more about the theory of relativity.

In the relativistic formalism the four spacetime coordinates are put together in the four-vector \( x^\mu = (x^0, x^1, x^2, x^3) \), where \( x^0 = ct, x^1 = x, x^2 = y, x^3 = z \); or, in abbreviated notation \( x^\mu = (ct, \vec{x}) \). Equation (31) represents then a set of four equations, one for each value of the Greek index \( \mu = 0, 1, 2, 3 \). The dots in equation (31) represent the operation of taking a time derivative and multiplying the result by the factor \( \gamma \) defined by

\[
\gamma = (1 - \beta^2)^{-1/2}
\]  

(33)

where \( \beta = v/c, v \) is the magnitude of the ordinary velocity \( v \) and \( c \) is the speed of light. Thus, \( \dot{x}^\mu := \dot{x}^\mu = (\gamma c, \gamma v) \). In addition, for the circular motion with constant angular speed that we are studying \( \gamma \) is constant in time, and therefore \( \dot{v}^\mu = (0, \gamma a) \) and \( \ddot{v}^\mu = (0, \gamma \frac{da}{dt}) \), where \( a \) is the acceleration vector of the electron.

The symbol \( F^{\mu\nu} \) that appears in (31) contains the external electric and magnetic fields acting on the electron, \( E \) and \( B \) respectively, in such a way that the combination \( F^{\mu\nu}v_\nu \) corresponds to \( \gamma v \cdot B/c, \gamma E + \gamma v \times B \). The effect of the self-fields of the electron on its movement, as well as the radiation emitted by the electron, is described by means of the term proportional to \( \tau_0 \) that appears in equation (31). Finally, for a given four-vector \( A^\mu = (A^0, \vec{A}) \) the repeated index notation \( A^\mu A_\mu \) denotes the combination \( A^\mu A_\mu = (A^2 - A^0)^2 \). Thus, for an electron in a circular orbit of radius \( a \) that moves with constant angular speed \( \omega \), we have \( \dot{v}^\mu \dot{v}_\mu = \gamma^4 \omega^2 a^2 \).

In figure 3 we have drawn the trajectory of the electron, which has been chosen to rotate counterclockwise with constant angular velocity \( \omega \) in a circular orbit of radius \( a \), larger than the solenoid’s radius \( b \). This movement constitutes an exact solution of equation (31) when the electron moves in the tangential electric field (11) and a homogeneous time-independent magnetic field that has an appropriate value and points along the solenoid’s axis (z-axis). Note that in what follows the only role of the magnetic field inside the solenoid is to provide the electric field outside the solenoid, where the motion takes place. In the Cartesian coordinate system presented in figure 3 we have

\[
\begin{align*}
E &= E \sin \omega t \hat{x} - E \cos \omega t \hat{y} \\
B &= B \hat{z} \\
x &= a \cos \omega t \hat{x} + a \sin \omega t \hat{y} \\
v &= -a \omega \sin \omega t \hat{x} + a \omega \cos \omega t \hat{y}; \quad \beta = a \omega / c \\
a &= -a \omega^2 \cos \omega t \hat{x} - a \omega^2 \sin \omega t \hat{y} \\
d\vec{a}/dt &= a \omega^3 \sin \omega t \hat{x} - a \omega^3 \cos \omega t \hat{y}.
\end{align*}
\]  

(34)

From these equations it follows that (31) is identically satisfied for \( \mu = 3 \) (or z-component), without imposing any restrictions on the magnitudes \( E \) and \( B \) of the electric and magnetic fields respectively. On the other hand, equation (31) for \( \mu = 1 \) (or x-component) becomes

\[
-(eE + m \tau_0 a \omega^3 \gamma^4) \sin \omega t = (ma \omega^2 \gamma + eB) \cos \omega t
\]  

(35)

while for \( \mu = 2 \) (or y-component) it becomes

\[
(eE + m \tau_0 a \omega^3 \gamma^4) \cos \omega t = (ma \omega^2 \gamma + eB) \sin \omega t.
\]  

(36)

Therefore, equation (31) for \( \mu = 1 \) and 2 is fulfilled for any time \( t \) if the magnitudes \( E \) and \( B \) of the fields are chosen as follows:

\[
E = -\frac{m \tau_0 a \omega^3 \gamma^4}{e}
\]  

(37)
Figure 3. Circular orbit of the electron, of radius $a$ larger than the solenoid’s radius $b$. The electron moves in a counterclockwise direction with angular velocity $\omega$ and speed $v = \omega a$. The electric field $E$ produced by the current in the solenoid compensates the radiation reaction force. We also show the radial and tangential unit vectors at the electron position, $\hat{\rho}$ and $\hat{\phi}$, respectively.

$$B = -\frac{mc^2 \beta \gamma}{ea}.$$  \hspace{1cm} (38)

Equation (31) for $\mu = 0$ is also fulfilled, since it reproduces (37) once again.

Although the field of the solenoid (18) allows one to construct the solution of equation (31) describing a monoenergetic electron in circular orbit, this does not mean that it has any practical interest. In this respect let us point out that the electric field that accelerates the electrons in circular accelerators, as well as the one that compensates the radiation reaction force in the storage rings, is very different from the one given in (18). The field generated by the infinitely long solenoid (18) is also of theoretical interest in constructing exact solutions of the equation of motion for more than one charge [5–7], as well as in constructing solutions of equations of motion more general than the Lorentz–Dirac equation (31) [8].

The electric field required by equation (37) can be obtained, for any orbital radius and for any electron velocity less than the velocity of light, by an appropriate choice of the parameter $\alpha$ in the current density (21). In contrast with the electric field (37), the magnetic field (38) does not depend on the parameter $\tau_0$ given in equation (32). For this reason the magnetic field (38) is exactly the same as that appearing in the case of the equation of motion of an electron in a pure homogeneous magnetic field when the effect of the radiation is not taken into account.

The counterbalance of the radiation reaction force by means of the tangential electric field of the solenoid can be clearly understood if (35) and (36) are written in terms of the radial and tangential unit vectors $\hat{\rho}$ and $\hat{\phi}$, respectively, as presented in figure 3. These vectors are given in terms of the Cartesian unit vectors $\hat{x}$ and $\hat{y}$ by means of

$$\hat{\rho} = \hat{x} \cos \omega t + \hat{y} \sin \omega t$$
$$\hat{\phi} = -\hat{x} \sin \omega t + \hat{y} \cos \omega t.$$  \hspace{1cm} (39)

Equations (35) and (36) then become

$$(eE + m\tau_0 \omega a^3 \gamma^4)\hat{\phi} = (m\omega^2 \gamma + e\beta B)\hat{\rho}.$$  \hspace{1cm} (40)

Now, the electric force on the electron is $eE$ and points along the tangent to the circle $\rho = a$ in the counterclockwise direction; therefore, the force due to the radiation reaction is $m\tau_0 \omega a^3 \gamma^4$, which, according to (37), exactly cancels the force due to the electric field. In other words, the tangential acceleration is zero, in agreement with the fact that the tangential speed $v = a\omega$ is constant.
Equation (37) also allows one to derive a quantitative expression for the rate of radiation emitted by the electron. To this end it is convenient to write (37) in the following form:

\[ e v \cdot E = \frac{e^2 c}{6\pi \varepsilon_0 a^2} \beta^4 \gamma^4. \]  

(41)

The left-hand side of this equation represents the power that the electric field \( E \) supplies to the electron. Now, since for a given orbit the speed of the electron remains fixed, the kinetic energy of the electron does not change. This means that all the energy supplied to the electron by the external electric field must be radiated away. In other words, a monoenergetic electron in a circular orbit emits radiation, that escapes to infinity, at a rate given by

\[ \frac{e^2 c}{6\pi \varepsilon_0 a^2} \beta^4 \gamma^4. \]

6. Conclusions

Textbooks usually calculate the magnetic field of an infinite solenoid fed with a time-independent surface current density per unit length only at points of its axis. In addition, some textbooks need to make further assumptions regarding the behaviour of the field outside the solenoid in order to obtain the correct result. Here we have shown how to obtain the magnetic field everywhere in a detailed way using the symmetries of the problem. Moreover, starting from this result it has been possible to guess the electric and magnetic fields produced by the solenoid when it carries a surface current density that increases linearly with time, thus providing one of the few elementary solutions of the Maxwell equations for time-dependent sources.

This solution has also been used to construct an exact solution of the Lorentz–Dirac equation, which is the relativistic equation of motion that takes into account the effect of the emission of radiation on the motion of the charge. The simplicity of this exact solution has made it possible to perform an energy balance that easily leads to an expression for the rate of radiation of a monoenergetic charge in circular motion. This discussion also provides a brief introduction to the Lorentz–Dirac equation and a motivation for inquisitive students to study the relativistic formalism further.

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