In this chapter we begin our study of more realistic systems in which the objects are no longer point particles but have extension in space. Up until now we’ve generally limited ourselves to the dynamics of point masses, first in one dimension and then generalized to two and three dimensions. Indeed, not all of the problems we studied were limited to point masses, but the object’s size and shape were not relevant in the problem and so were not considered. For such objects we’ve learned how to describe and predict translational motion using Newton’s laws, some of the complications due to frictional forces, and the important concept of energy. In general we can divide the motion of real extended bodies into two parts: translational motion, described by following a particular average coordinate of the object, known as its center of mass as it moves about, and all other motions with respect to this point. This chapter focuses on translational motion of systems, or collections of objects, and the following chapter deals with rotational motion.

We begin this chapter by introducing the important concept of momentum. As we’ve seen, all forces come in pairwise interactions. When studying the interactions between different objects, it turns out that we can re-formulate Newton’s second law in terms of momentum. If the system we are studying is “isolated”—meaning that it does not interact with the outside world—then our reformulation is particularly simple and leads to a new fundamental law, the law of conservation of momentum. After seeing this for a system of two particles, we next define and learn how to compute the center of mass of a system, that special average point of a system at which all its mass appears to be concentrated in order to explain the net translational motion of the system. The last section of the chapter shows how to reformulate the dynamics of translational motion of any system in terms of the center of mass momentum. Here we also see the general formulation of conservation of momentum.

1. Momentum

Thus far in our discussions of dynamics we have focused on forces as the origin of motion according to Newton’s laws. There is a very useful alternative approach based on momentum that we wish to develop in this chapter. Very often this alternative approach is to be preferred because it does not hinge on the specific forces or interactions between objects, which are usually unknown or only incompletely understood. In this section, we first introduce momentum, the basic quantity used in this approach, for a particle. Then we reformulate Newton’s second law using momentum and show how this leads to the conservation of momentum principle for a collection of particles. Later in this chapter we generalize this approach to arbitrary collections of extended objects.

An object of mass $m$ traveling at velocity $\vec{v}$ has a linear momentum (or just momentum) $\vec{p}$, given by

$$\vec{p} = m\vec{v}. \quad (6.1)$$
Newton’s second law for an object can be written in terms of momentum as

\[
\vec{F}_\text{net} = \frac{d\vec{p}}{dt}.
\]

Using the definition of \( \vec{p} \) (Equation (6.1)) and the product rule for derivatives, we can write this as

\[
\vec{F}_\text{net} = \frac{d(m\vec{v})}{dt} = m \frac{d\vec{v}}{dt} + \vec{v} \frac{dm}{dt}.
\]

In the case when the mass is not changing the last term vanishes and using the definition of acceleration we get the usual form of \( \vec{F}_\text{net} = m\vec{a} \). In cases where the mass is changing (e.g., a rocket ejecting substantial amounts of fuel), the full expression is needed and this form of Newton’s second law is the correct expression.

Note that momentum is a vector quantity, defined as the product of the mass, an intrinsic property of the object, and its velocity, a quantity depending on its motion. It has units of kg-m/s, which have no other special name. Clearly, based on Newton’s first law, a particle with no net force on it will maintain a constant momentum. When the particle feels a net force, due to some interaction, its momentum will change with time. Also clearly, based on Newton’s second law, the larger the interaction (force) acting on the particle, the greater will be the change in its momentum.

How does the momentum of a particle contrast with its velocity? First, we note that both of these quantities are vectors, in fact with the same direction. If we compare two particles of different mass traveling at the same velocity, the one with larger mass will also have proportionally larger momentum. For example, a truck with four times the mass of a car, both traveling at the same speed along a highway, has four times the momentum of the car, in accord with our colloquial usage of the word momentum. On the other hand, if the same truck is traveling at only 1/4 the velocity of the car, then both vehicles have the same momentum.

How does the momentum of a particle contrast with its kinetic energy? Now, note that these are very different quantities, with kinetic energy a scalar and momentum a vector. A particle with a given mass will have its momentum doubled if its velocity doubles, but will have its kinetic energy quadrupled in that case. Kinetic energy is produced by doing work on a particle, as we’ve seen in the work–kinetic energy theorem. How is momentum produced? Well, clearly they are related, but the direct answer is that momentum is produced by forces acting on the particle as we now show.

Newton’s second law for an object can be written in terms of its momentum by noting that \( m\vec{a} \) is defined as

\[
m = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t},
\]

and because \( m \) is constant, we can further write that

\[
\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}}{\Delta t}.
\]

We therefore find that

\[
\vec{F}_\text{net} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}}{\Delta t}.
\]

This is actually the form that Newton proposed for the second law and is more general than the form \( \vec{F}_\text{net} = m\vec{a} \), because it allows for cases in which the mass of an object may change with time. Such a situation might arise when mass is either being added or removed from the object over time (see the boxed discussion). For example, a rocket burns fuel and decreases its mass by ejecting the waste gases or the mass of a boat that is drifting by a pier may suddenly increase when you jump into it. In both of these cases our original form of \( F = ma \) does not apply because the mass is changing.

**Example 6.1** An *E. coli* bacterium of mass \( m = 6 \times 10^{-16} \) kg is initially swimming at a constant velocity of 8 \( \mu \)m/s toward the east. One ms later it is found to be swimming at 10 \( \mu \)m/s toward the north. Find the change in the *E. coli*’s momentum and the average external force acting on the bacteria during the 1 ms time interval.
Solution: It is very tempting to write that the change in the bacterium’s momentum is the product of its mass and the change in its speed \(10 - 8 = 2 \text{ \mu m/s}\). This temptation must be strongly resisted because it is the change in the velocity vector that is appropriate and this is not a one-dimensional problem. Figure 6.1 shows a vector diagram for the initial and final momenta and the change in momentum of the bacterium. From the figure it is clear that the change in momentum is found from the hypotenuse of the triangle formed so that

\[
\Delta p = m \sqrt{v_{ini}^2 + v_{fin}^2} = 6 \times 10^{-16} \sqrt{(8 \times 10^{-6})^2 + (10 \times 10^{-6})^2} = 7.7 \times 10^{-27} \text{ kg \cdot m/s}
\]

The direction of this momentum change is given by

\[
\theta = \tan^{-1}\left(\frac{8}{10}\right) = 39^\circ
\]

where the angle \(\theta\) is measured west of north as shown in the figure.

The average force acting over this interval of time is then given by Equation (6.2) (without the limit) and is found to be

\[
\bar{F} = \frac{\Delta \bar{p}}{\Delta t} = \frac{7.7 \times 10^{-27}}{10^{-3}} = 7.7 \times 10^{-24} \text{ N}
\]

in the same direction as the momentum change.

Example 6.2 The fastest passenger elevator in the world (in a 70-story building in Yokohama, Japan) attains a maximum speed of 12.5 m/s (28 mph) taking passengers from the ground to the top floor in 40 s. Find the maximum change in your momentum if you were to ride in this elevator. What is the net change in your momentum for the entire trip?

Solution: The maximum change in your momentum would occur during the acceleration or deceleration phase of the ride. Assuming your mass to be 80 kg, during the acceleration phase your momentum would increase from zero to \(p = (80 \text{ kg}) (12.5 \text{ m/s}) = 1000 \text{ kg \cdot m/s}\), so that your maximum change in momentum would just be 1000 kg m/s. For the entire trip to the 70th floor your net change in momentum is zero because both your starting and ending momentum are zero.

Suppose that two otherwise isolated point particles undergo a collision. We would like to understand what occurs and be able to predict the outcome. When far enough apart, the two particles move independently and do not interact. They will each have some momentum and if they are to collide must be moving along a line connecting them; let’s call this the x-axis and we see that this problem for two-point particles is really one-dimensional. Momentum is a particularly useful concept in this situation, as we show. Suppose that particle #1 has momentum \(p_1\) and particle #2 has momentum \(p_2\), both directed along the x-axis. For them to collide they must be moving toward each other, but they might both be moving in the same direction with one
particle “catching up” to the other, so let’s label the momenta as both positive for this discussion.

If we write Equation (6.2) for each of the particles we have

\[ \vec{F}_{2on1} = \frac{\Delta \vec{p}_1}{\Delta t} \quad \text{and} \quad \vec{F}_{1on2} = \frac{\Delta \vec{p}_2}{\Delta t}, \] (6.3)

where the only force on each particle is from the other one. These forces need not be contact forces acting only during a (macroscopic) contact between the two particles, but can also be long-range forces acting over long distances. Now, using Newton’s third law, we know that these two forces are reaction-pair forces and are always equal and opposite to each other. We can conclude then that because the vector sum of the two forces always adds to zero, we must have at all times that

\[ \frac{\Delta \vec{p}_1}{\Delta t} + \frac{\Delta \vec{p}_2}{\Delta t} = 0 \quad \text{or} \quad \frac{\Delta (\vec{p}_1 + \vec{p}_2)}{\Delta t} = 0 \quad \text{or} \quad \Delta (\vec{p}_1 + p_2) = 0. \] (6.4)

For this to be true it must be that the net momentum of the two particles remains constant with time. We say that, in this situation, momentum is conserved. We have specifically written these last few steps using vectors and in a general way to show the power of the law of conservation of momentum, even though our current example is one-dimensional. All we have used in this derivation are Newton’s laws (specifically the second law written in terms of momentum and the third law) and the fact that the particles were otherwise isolated, not interacting with any other objects. Thus, we really have proven that any two isolated objects, not necessarily point particles, that collide will have a total momentum that remains constant (Figure 6.2). Furthermore, even if there are external forces acting on the two particles, as long as there are no external forces acting along the direction of their motion, momentum will still be conserved. For example, momentum will be conserved for horizontal (frictionless) motion of two colliding objects even though gravity may act vertically. We show this in a couple of examples just below.

What does this tell us about the interactions between the two particles and the outcome of the collision? The beauty of this formulation is that the outcome is independent of the interactions; we do not need to know anything about the details of the interaction in order to predict something about the outcome. All we need to know is contained in Equations (6.3) and (6.4). During the collision the two objects will exert equal and opposite forces on each other for some period of time. If the collision involves short-range forces, so that the collision time \(\Delta t\) is short, then the product of the (typically) large force on one particle from the other and the short collision time is called the impulse,

\[ \text{Impulse} = F\Delta t = \Delta p = p_{\text{final}} - p_{\text{initial}}. \] (6.5)
The impulse represents the “net effect” of a collision between two objects. It lumps together the acting force and its duration into a single parameter that is able to predict the change in momentum of the particle due to the collision. Figure 6.3 shows a plot of a typical interaction force on a particle as a function of time during a collision. The impulse represents the area under this curve, equal to the average force acting multiplied by the duration $\Delta t$.

Suppose, for example, the two objects are identical, with the same mass $m$, and are traveling toward each other at the same speed $v$. Then, although each object has momentum $mv$, the net momentum before the collision is, in fact, zero. Do you see why? (Remember that momentum is a vector!) In this case, conservation of momentum predicts that the final momentum must be zero as well. There are two possible final situations for which the final momentum can be zero. In one case the two particles stick together and come to rest, whereas in the other case they bounce off each other and go off in opposite directions with the same magnitude of momentum that they had, and thus at the same speed. Although both of these situations conserve momentum, they differ in whether they conserve kinetic energy. The two particles that stick together and come to rest clearly have lost their kinetic energy, giving it up to other forms of energy such as sound and heat, because we know that ultimately energy must be conserved.

In more complex situations with two unequal mass objects traveling at different speeds, the algebra becomes a bit more involved and the possible outcomes will depend on whether kinetic energy is conserved. We do not dwell on these situations in detail, but simply point out that conservation of momentum offers a major additional tool in their study. In the third section of this chapter we generalize our formulation of conservation of momentum to more complex systems. A few examples should help you to appreciate the power of this new conservation law.

**Example 6.3** A 60 kg boy dives horizontally with a speed of 2 m/s from a 100 kg rowboat at rest in a lake. Ignoring the frictional forces of the water, what is the recoil velocity of the boat?

**Solution:** Since there are no external horizontal forces acting (we have ignored the frictional resistance force of the water here), momentum is conserved as the boy dives off the boat. Because the initial momentum of the (boy + boat) system is zero, the total momentum immediately after the boy dives off the boat must also be zero so that the boy and the boat must have equal, but oppositely directed, momenta. Note that it is the momenta that must be equal and opposite, not the velocities. In equation form

$$\vec{P}_{\text{ini}} = \vec{P}_{\text{fin}}, \text{ with } \vec{P}_{\text{ini}} = 0 \text{ and } \vec{P}_{\text{fin}} = \vec{P}_{\text{boy}} + \vec{P}_{\text{boat}}.$$

We therefore have that

$$0 = (60 \text{ kg}) (2 \text{ m/s}) + (100 \text{ kg}) v_{\text{boat}}.$$

so that the boat’s recoil velocity is found to be 1.2 m/s in the direction opposite to the boy’s velocity.

**Example 6.4** Two ice skaters, both traveling at a speed of 5 m/s and heading straight toward each other, collide and lock arms together. If their masses are 80 g and 50 kg, find the velocity with which they move together after the collision.  

(Continued)
In Example 6.3 we ignored the fluid medium and its frictional force. The surrounding fluid medium is often of primary importance. Let’s turn our attention to the problem of animal locomotion and, in particular, the motion of sea creatures such as the squid or jellyfish. These creatures, and indeed all animals that swim or fly through a fluid medium, move by virtue of reaction forces provided by the surrounding fluid medium. The jellyfish propels itself by jet propulsion, ejecting a volume of water in a jet that provides a thrust force in the opposite direction. Fish and birds generate thrust in a more continuous fashion by pushing back on the fluid medium with fins or wings (Figure 6.4). In any case, we can analyze such locomotion in either of two ways: a difficult method using the detailed reaction forces or much more easily using momentum.

Let’s discuss the jet propulsion of a jellyfish in order to derive an expression for the thrust propelling it. We can model the jellyfish as a balloon that fills with water and then collapses driving water out in a jet (Figure 6.5). Let the initial mass of water contained within the balloon be \( m_0 \) and suppose that the collapse results in a uniform rate of decrease of the mass, \( \frac{\Delta m}{\Delta t} \). Then the rate at which momentum is ejected from the balloon will be

\[
\frac{\Delta p}{\Delta t} = \frac{\Delta m}{\Delta t} v,
\]

where we assume a constant velocity for the jet of water expelled. By Newton’s second law, the rate of momentum ejection provides a net force, known in this context as the thrust. If we take the initial volume of the jellyfish to be that of a 0.1 m radius sphere filled with water of density \( \rho = 1000 \text{ kg/m}^3 \), then \( m_0 = \text{(volume)} \times \text{(density)} = \frac{4}{3} \pi r^3 \rho = 4.2 \text{ kg} \) of water. If this water is ejected in 1 s through a 1 cm radius circular

\[
\text{Solution:} \quad \text{There are no horizontal external forces acting, so therefore momentum is conserved and we know that the sum of the skaters’ two initial momenta is equal to their combined final momentum. Initially their momentum is } P_{\text{ini}} = (80 \text{ kg}) (5 \text{ m/s}) - (50 \text{ kg}) (5 \text{ m/s}) = 150 \text{ kg-m/s in the direction the 80 kg skater is traveling. When they lock together, their combined mass is 130 kg and we must have that } P_{\text{fin}} = (130 \text{ kg}) v_{\text{fin}} = P_{\text{ini}} = 150 \text{ kg-m/s}, \text{ so that } v_{\text{fin}} = 1.2 \text{ m/s in the direction the heavier skater was traveling.}
\]

In Figure 6.4 we illustrate that, bird, rocket, or fish, propulsion is by thrust, a reaction force, that conserves momentum.
aperture, then we can first calculate the velocity of water flow. This is found by assuming that the total volume of water flows out in a cylindrical jet whose length is proportional to the velocity; that is, \( \rho L = m_0 \), where \( L = vt \). Knowing the mass \( m_0 \), density, cross-sectional area \( A = \pi r^2 \) (0.01 m), and time \( t = 1 \) s, we can calculate the water velocity to be \( v = 13 \) m/s first, and then

\[
F = \frac{\Delta m}{\Delta t} v
\]

to find about 55 N of thrust is generated. This is actually a greater force than the weight of the initial water contained in the balloon. A similar analysis can explain the thrust of a rocket or that of a bird, but a realistic analysis will be more complex because of the nonconstant velocities involved in the problem.

2. CENTER OF MASS

The simplest systems are composed of a single point particle introduced in the previous chapters. Here we begin our systematic study of increasingly more complicated systems of two particles, of many particles, or of a single extended object such as a person. For any system there is a well-defined point, the center of mass, at which the entire mass of the system can be considered to be concentrated in order to understand its translational motion.

A rough analogy to finding the center of mass can be made to locating the population center of the United States. Rather than weighting locations by their mass, they are weighted, in this case, by their local populations. This two-dimensional problem on a map could be attacked in a number of approximate ways, one of which we illustrate. Using census figures for the state populations and choosing some appropriate location as the population center within each state (e.g., by specifying latitude and longitude of its largest city) one could find the U.S. population center by separately averaging the latitude and longitude of the states, weighting each by its population. Thus California, Texas, and New York, together with more than one third of the U.S. population, dominate in the calculations and we expect to find a population center somewhere in the Midwest, even though the Midwest population is not particularly large. This example illustrates the notion of weighting locations by a local property or characteristic.

To introduce the definition of center of mass consider two particles of masses \( m_1 \) and \( m_2 \) attached by a light (massless) rod of length \( L \) as shown in Figure 6.6. If this system were tossed into the air it would translate and rotate about before landing on the ground. One special point, the center of mass of the system, would travel in the same trajectory as a single particle of mass \( (m_1 + m_2) \) launched with the same initial velocity (we show this in the next section). Qualitatively this point can be imagined to be determined by finding the balance point along the rod. That is, imagine moving your finger along the rod until you can balance the rod with its masses on either end. That point is also the center of mass. For example, if \( m_1 = m_2 \) the center of mass would be located in the center of the rod at a distance of \( L/2 \) from either end. If \( m_1 > m_2 \), then the balance point would be closer to \( m_1 \), but how much closer?

Because the balance point in Figure 6.6 will be closer to the more massive particle, we want to define the center of mass as an average position of the two particles, with more massive particles counting more in the averaging process. We therefore define the center of mass along one dimension, \( x_{cm} \), relative to an arbitrary origin, as
We can generalize this definition in a straightforward way to systems of more
than two particles by simply adding terms for additional point masses in both the
numerator and denominator in Equation (6.6). However, with more than two parti-
cles, the system need not be one-dimensional if the masses are not co-linear. In this
case we can also define the $y$- (and $z$, if needed) components of the center of mass
in a similar way and combine them by writing $x_{cm}$.

\[
x_{cm} = \left( \frac{m_1}{m_1 + m_2} \right) x_1 + \left( \frac{m_2}{m_1 + m_2} \right) x_2, \tag{6.6}
\]

where $x_1$ and $x_2$ are the $x$-coordinates of masses $m_1$ and $m_2$, respectively.

**Example 6.5** Find the center of mass of the Earth–moon system given that
the mean radius of the Earth is $6.37 \times 10^6$ m, the mean radius of the moon is
$1.74 \times 10^6$ m, the Earth–moon mean separation distance is $3.82 \times 10^8$ m, and
that the Earth is 81.5 times more massive than the moon.

**Solution:** The Earth–moon separation is so much larger than the radius of either;
therefore we can treat both bodies as point masses for the purposes of this cal-
culation. With an origin at the center of the Earth (see Figure 6.7), we can write

\[
x_{cm} = \frac{M_e (0) + M_m (L)}{M_e + M_m} = \frac{1}{1 + \frac{M_e}{M_m}} L = 0.012 \times 10^6 \text{ m}.
\]

Thus, the center of mass of the Earth–moon system actually lies within the
Earth.

We can generalize this definition in a straightforward way to systems of more
than two particles by simply adding terms for additional point masses in both the
numerator and denominator in Equation (6.6). However, with more than two parti-
cles, the system need not be one-dimensional if the masses are not co-linear. In this
case we can also define the $y$- (and $z$, if needed) components of the center of mass
in a similar way and combine them by writing $r_{cm} = (x_{cm}, y_{cm}, z_{cm})$. Using the
summation notation that $\sum$ indicates to sum over all particles in the system, we can write (with a similar equation for $z_{cm}$)

\[
x_{cm} = \sum \left( \frac{m_i}{M} \right) x_i \quad \text{and} \quad y_{cm} = \sum \left( \frac{m_i}{M} \right) y_i, \tag{6.7}
\]

where $M$ is the total mass of the system. The subscript $i$ denotes a particular numbered
particle and the summation sign indicates that $i$ is to be varied from number 1 to the
total number of particles in the system while performing the additions indicated.

**Example 6.6** Find the center of mass of a water molecule using the following
data (Figure 6.8): radius of O = 0.14 nm, radius of H = 0.12 nm, bond length of
O–H bond = 0.097 nm, and H–H angle subtended at O = 104.5°.
Solution: We solve this problem in two different ways using two different coordinate origins to see that the answer is independent of the chosen origin.

(1): In the first solution we set the origin on the O center and use the axes shown on the left. In this case the atoms have their centers located at: O (0,0); H \((-0.097 \cos 52.3^\circ, \pm 0.097 \sin 52.3^\circ) = (-0.059, \pm 0.077)\), where the O–H bond is the center-to-center distance and we have found the \(x\)- and \(y\)-components of the H centers. We solve for the \(x\)- and \(y\)-coordinates of the center of mass (taking the masses of O and H as 16 and 1) by writing

\[
x_{\text{cm}} = \frac{16(0) + 1(-0.059) + 1(-0.059)}{16 + 1 + 1} = -0.0066 \text{ nm},
\]

and

\[
y_{\text{cm}} = \frac{16(0) + 1(0.077) + 1(-0.077)}{18} = 0,
\]

where the zero value for \(y_{\text{cm}}\) should be expected from the fact that the two H atoms are symmetrically situated above and below the \(x\)-axis.

(2): Using the coordinates shown on the right in the figure the atoms have their centers located at: O \((0.097 \cos 52.3^\circ, 0.097 \sin 52.3^\circ) = (0.059, 0.077)\), H \((0, 0)\) and H \((0, 2 \cdot 0.097 \sin 52.3^\circ) = (0, 0.15)\). Using the same basic relations we write

\[
x_{\text{cm}} = \frac{16(0.059) + 1(0) + 1(0)}{18} = 0.053 \text{ nm},
\]

and

\[
y_{\text{cm}} = \frac{16(0.077) + 1(0) + 1(0.15)}{18} = 0.077 \text{ nm}.
\]

Although these two answers appear at first glance to be different, the shift in origins must be accounted for in comparing them. The origin on the right is located at the point \((-0.059, -0.077)\) with respect to the origin on the left and if we compare the actual spatial location of the center of mass in both parts (by, e.g., adding the origin coordinates on the right with respect to those on the left

(Continued)
In the case of a solid extended object, if it is uniform throughout and has some symmetry we can often determine its center of mass by inspection, based on the notion of a balance point that was qualitatively introduced with Figure 6.6. For example, a uniform solid rod will have its balance point, or center of mass, at its geometric center. Even if an object has multiple parts, each of which is uniform throughout and has some symmetry, we can reduce the problem to finding the center of mass of a collection of particles, one for each part of the object with the mass of each part located at the center of mass of that part. This is illustrated in the following example.

A different approach to the problem might be to first recognize that by symmetry the $y_{cm}$ must lie along the $x$-axis using the left set of coordinates in Figure 6.8 and then to set up the problem as a two-mass system with the total H mass located at $x = -0.059$ nm and the O mass at the origin. Try it.

In the case of a solid extended object, if it is uniform throughout and has some symmetry we can often determine its center of mass by inspection, based on the notion of a balance point that was qualitatively introduced with Figure 6.6. For example, a uniform solid rod will have its balance point, or center of mass, at its geometric center. Even if an object has multiple parts, each of which is uniform throughout and has some symmetry, we can reduce the problem to finding the center of mass of a collection of particles, one for each part of the object with the mass of each part located at the center of mass of that part. This is illustrated in the following example.

**Example 6.7** The solid objects shown in the three figures below are all made from the same uniform material and have the same thickness. In part (c) there is a small hole in the larger circular plate. Find the center of mass of each object using the coordinate system shown. Take $R = 0.1$ m and $L = 0.05$ m.

**Solution:** Because all the objects are made of a uniform material and have the same thickness, their masses are simply proportional to their areas. This is true because the mass $m$ is equal to the product of the density $\rho$ of the material, its thickness $t$, and its area $A$, or

$$m = \rho \, t \, A.$$

Because both the density and thickness are constants, $m \propto A$, and furthermore, because in Equation (6.6) only the ratio of masses appears, we do not need to know the thickness or density of the materials as they cancel and do not appear in the final result. In what follows we therefore set the proportionality constant simply equal to 1 and numerically equate masses and areas.

We proceed by replacing each regular shape with a point mass having the same total mass as that portion of the entire object and located at its center of mass (these are found by inspection because the shapes are highly symmetric). Each problem then reduces to a set of point masses, all located in the same plane. In part (c) we use a trick: let the hole have a negative mass according to its size and superimpose the larger solid circular plate with the
smaller circular plate of “negative” mass, thus canceling the mass within the hole region!

In (a) we have the following three objects: \( M = 4R^2 = 0.04 \) located at \( (R, 3R) = (0.1, 0.3) \); \( M = \pi R^2 = 0.031 \) at \( (R, R) = (0.1, 0.1) \); and \( M = \pi R^2 = 0.031 \) at \( (3R, 3R) = (0.3, 0.3) \). We then find that

\[
x_{cm} = \frac{4R^2(R) + \pi R^2(R + 3R)}{4R^2 + 2\pi R^2} = 0.16 \text{ m},
\]

and

\[
y_{cm} = \frac{4R^2(3R) + \pi R^2(R + 3R)}{4R^2 + 2\pi R^2} = 0.24 \text{ m}.
\]

In (b) we have three point masses: \( M = 2L^2 = 0.005 \) at \( (L/2, L) = (0.025, 0.05) \); \( M = 4L^2 = 0.01 \) at \( (2L, 2.5L) = (0.1, 0.125) \); and \( M = 3L^2 = 0.0075 \) at \( (2L, 4.5L) = (0.1, 0.225) \). The center of mass is given by

\[
x_{cm} = \frac{2L^2(L/2) + 4L^2(2L) + 3L^2(2L)}{9L^2} = 0.083 \text{ m},
\]

and

\[
y_{cm} = \frac{2L^2(L) + 4L^2(2.5L) + 3L^2(4.5L)}{9L^2} = 0.14 \text{ m}.
\]

In (c), using the trick mentioned above, we have two point masses: \( M = \pi(2R)^2 = 0.13 \) at \( (2R, 2R) = (0.2, 0.2) \); and \( M = -\pi(R/2)^2 = -0.0079 \) at \( (2R, 3R) = (0.2, 0.3) \). Using the same method we find

\[
x_{cm} = \frac{4\pi R^2(2R) - \pi \frac{R^2}{4}(2R)}{3.75\pi R^2} = 0.2 \text{ m},
\]

as expected, and

\[
y_{cm} = \frac{4\pi R^2(2R) - \pi \frac{R^2}{4}(3R)}{3.75\pi R^2} = 0.19 \text{ m}.
\]

This last number seems reasonable because with a hole cut out we expect \( y_{cm} \) to be somewhat less than \( 2R = 0.2 \) m.

You should go through each answer, following all the steps, and see that the center of mass position makes qualitative sense.

In the general case of nonuniform and/or nonsymmetric objects, the center of mass can always be found experimentally by finding the balance point along three mutually perpendicular axes, if possible, or by suspending the object separately from three different points, drawing vertical lines from those points, and looking for the intersection of the three lines (Figure 6.10). The reason why this latter method works becomes clearer after we have discussed rotational motion, but the center of mass must lie suspended vertically under the suspension point. Alternatively, one can use more complex mathematics to calculate the center of mass position. We show a more
direct experimental approach to finding the center of mass in the next section. There we show that the center of mass translates about in space as if all external forces act directly on the entire mass of the system located at its center of mass.

3. CENTER OF MASS MOTION: NEWTON’S SECOND LAW AND CONSERVATION OF MOMENTUM

In the last section we learned how, in principle, to find the center of mass of any object, and in practice, to find that point for a collection of particle masses or symmetric objects. Here we show that the translational motion of a system of particles or an extended object is fully described by knowledge of the center of mass motion. The main goals of this section are to generalize Newton’s second law for a particle to a very similar result for the center of mass of a system and to generalize the law of conservation of momentum.

The derivation of the generalization of Newton’s second law to a system of extended objects is straightforward using some calculus (see box on the next page), but otherwise is cumbersome. The resulting *Newton’s second law for a system* is

\[
\vec{F}_{\text{net, ext}} = \sum \vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}},
\]

where the sum \( \sum \) is over all of the external forces acting on the system, \( M \) is the total mass of the system (assumed constant; a system that does not exchange mass with its surroundings is known as a *closed system*), and \( \vec{a}_{\text{cm}} \) is the acceleration of the center of mass. In this expression the only forces that produce an acceleration of the center of mass are forces exerted on the system by objects that are external to the system, so-called *external forces*. All of the *internal forces* between the particles of the system cancel pairwise because they are equal and opposite according to Newton’s third law. This equation also applies to extended objects because they can be considered to be built up from particles. We conclude that the translational motion of a system can be completely described by replacing the entire system by a point mass with total mass \( M \) located at the system’s center of mass, \( \vec{r}_{\text{cm}} \), with only external forces acting (Figure 6.11).
As a byproduct of the derivation of Equation (6.8), we show (in the box) that the total momentum of the system, the vector sum of the individual particle momenta, is equal to the momentum of the center of mass, or the product of the total mass $M$ and the center of mass velocity $v_{cm}$.

$$\vec{P}_{cm} = M\vec{v}_{cm} = \sum \vec{p}_i.$$  

(6.9)

Thus an alternative way to write Equation (6.8), in terms of the center of mass momentum, is

$$\vec{F}_{\text{net, ext}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{P}_{cm}}{\Delta t}.$$  

(6.10)

We see that the center of mass moves as if all the mass of the system is located there and experiences the net external force on the entire system. So, no matter whether the system is composed of a single extended object (such as the high jumper of Figure 6.11) or many independent parts (such as the exploded rocket in that figure), the center of mass of the system moves in a well-defined trajectory based on the total mass and the net external force on the system.

Written in this form we can deduce a very important consequence:

**In the absence of a net external force on a system, the center of mass momentum, or total momentum of the system, does not change with time, and is said to be conserved.**

This is a statement of the principle of conservation of momentum, a very powerful and general result, which holds for all isolated systems, those with no net external forces applied. It is a fundamental principle that holds on every scale of distance: on the atomic or nuclear scale as well as on the scale of the size of the universe. We saw a preliminary version of this in the first section of this chapter for collisions between two particles, but the principle is much more general than we saw there.

Conservation of momentum is the second of a handful of conservation laws that we study in this book. We have already learned the conservation of energy principle and seen its tremendous value as a tool in understanding motion. Later on we demonstrate its value in all other areas of physics that we study. Energy and momentum conservation are two of the cornerstones of physics. Because the momentum of an isolated system is constant, if we compute the total momentum at any time, its value at any other time will be the same vector, namely the same value and in the same direction. Just as with energy conservation, we can use our knowledge of the situation at one instant of time to find the total momentum, which will remain constant as long as there are no external forces acting. On the other hand, unlike energy conservation, momentum is a vector and therefore a direction as well as a magnitude is fixed in time.

We note that the kinetic energy of a particle $KE = \frac{1}{2}mv^2$ can in fact be rewritten in terms of the particle’s momentum in place of its velocity. Using the definition of the magnitude of the momentum $p = mv$, we have that $KE = p^2/2m$. You need to keep in mind that although the kinetic energy, a scalar, can be written in terms of the square of the particle’s momentum, the conservation laws of energy and of momentum are two different laws that keep different quantities constant. For a closed system (one with no exchange of mass with its surroundings) with no external forces acting, the total, or center of mass, momentum will be conserved as will the total mechanical energy. However, the kinetic energy of the system may change because it can be exchanged for potential energy and other forms.

**A derivation of Equation (6.8):** Starting from a rewriting of Equation (6.6) for the $x$-component of the center of mass

$$M\vec{x}_{cm} = \sum m_i \vec{x}_i,$$

we can differentiate both sides of the equation with respect to time to find

$$M\vec{v}_{cm} = \sum m_i \vec{v}_i,$$

or

$$\vec{F}_{cm} = \sum \vec{F}_i,$$

where $\vec{P}_{cm}$ and $\vec{p}_i$ are the momentum of the center of mass and the individual particles. Thus we see that the center of mass momentum is equal to the total momentum of the system of particles. If we further differentiate this equation with respect to time we have

$$M\vec{a}_{cm} = \frac{d\vec{P}_{cm}}{dt} = \sum \frac{d\vec{p}_i}{dt} = \sum \vec{F}_{i,\text{net}},$$

where we have used Newton’s second law for each particle, assumed that the total mass of the system is constant, and $\vec{F}_{i,\text{net}}$ is the net force on the $i$th particle. To complete the derivation of Equation (6.8), we note that the forces on particle $i$ are of two types: external, arising from objects outside the system, and internal, arising from other particles in the system. These latter internal forces cancel pairwise in the summation because a force on particle 2 from particle 3 is equal and opposite to the force on particle 3 from particle 2 and all possible pairs of forces will be summed. The final result is Equation (6.8). Newton’s law can be generalized still further to a system of extended objects with no change to Equation (6.8).
energy within the system. In the first section of this chapter we saw a few examples of the application of momentum conservation to the collision between two objects. Two quite different examples should help to provide an appreciation for the power of the conservation of momentum principle.

**Example 6.8** A rocket of mass $M$ explodes into three pieces at the top of its trajectory where it had been traveling horizontally at a speed $v = 10$ m/s at the moment of the explosion. If one fragment of mass $0.25M$ falls vertically at a speed of $v_1 = 1.2$ m/s, a second fragment of mass $0.5M$ continues in the original direction, and the third fragment exits in the forward direction at a $45^\circ$ angle above the horizontal (see Figure 6.12), find the final velocities of the second and third fragments. Also compare the initial and final kinetic energies to see how much was lost or gained.

**Solution:** Although the rocket is not an isolated system, the forces in the explosion are assumed to be so much greater than the weight of the rocket that we can neglect gravity at the moment of the explosion. This situation is very similar to that of any collision in which two objects interact very strongly for a very short time as, for example, when a tennis racket hits a ball. In all such cases we can neglect gravity during the collision and treat the system as isolated. Therefore the initial momentum of the rocket $P_{ini} = Mv$ in the horizontal direction must be conserved during the explosion and the sum of the momenta of the three fragments must add up to exactly this same $P_{ini}$ value. Using vector addition, we can write that conservation of momentum in the horizontal direction implies

$$Mv = \frac{1}{2}Mv_2 + \frac{1}{4}Mv_3 \cos 45^\circ,$$

where the velocities are labeled as in the figure. Note that the first fragment falls vertically and does not contribute to this equation for the horizontal momenta. Conservation of momentum in the vertical direction gives a second equation

$$0 = \frac{1}{4}Mv_1 - \frac{1}{4}Mv_3 \sin 45^\circ.$$

Substituting that $v_1 = 1.2$ m/s, we find first, from the second equation above after canceling the common factor $M$, that

$$v_3 = \frac{1.2}{\sin 45^\circ} = 1.7 \text{ m/s},$$

and then, on substitution into the first equation above, that

$$10 = 0.5v_2 + 0.25 \cdot 1.7\cos 45^\circ,$$

so that $v_2 = 19$ m/s.

The initial kinetic energy is $\text{KE}_i = \frac{1}{2}mv^2 = 50 M$, and the total final kinetic energy is $\text{KE}_f = \frac{1}{2}(M/4)(1.2)^2 + \frac{1}{2}(M/2)(19)^2 + \frac{1}{2}(M/4)(1.7)^2 = 90.8 M$, both measured in J with $M$ in kg. The kinetic energy has increased by over 80% with the
excess coming from the chemical energy released in the explosion. Kinetic energy alone is not conserved in this example, but momentum is. On the other hand, the general principle of conservation of energy is obeyed with the total mechanical, chemical, and other sources of energy remaining constant for the rocket.

In the above example we’ve seen how in an explosion we can use conservation of momentum to learn about the motion of the final pieces. Similarly in a collision between objects we can use conservation of momentum during the collision to learn about the final motions of the objects after collision. For microscopic objects that interact during a collision, of say atoms, the forces are all conservative and the collisions, aside from conserving momentum, tend to be elastic, conserving energy as well. In most cases of macroscopic objects colliding, the collisions tend to be inelastic, so that energy is lost (or gained in the explosion of the last example) even though momentum is conserved during the collision. The next two examples illustrate some of these possibilities.

Example 6.9 A hockey puck of 0.5 kg mass traveling at a speed of 5 m/s collides with an identical stationary puck in a glancing (not head-on) collision. If the first puck is deflected by 30° and travels with a final speed of 3 m/s, find the final velocity of the puck that was hit if it moves off at a 45° angle as shown. Ignore any friction between the ice and pucks.

Solution: Because there are no external horizontal forces acting, momentum is conserved. With the initial direction of motion chosen as the \(x\)-axis, the initial momentum is only

\[ p_{ix} = mv_0 = (0.5)(5) = 2.5 \text{ kg m/s}. \]

After the collision, both pucks have \(x\) momentum (components of their momentum vectors) that must add up to the initial momentum as

\[ 2.5 \text{ kg m/s} = (0.5 \text{ kg})(3 \cos 30 \text{ m/s}) + (0.5 \text{ kg})(v \cos 45), \]

where \(v\) is the final velocity of the second puck. Solving for \(v\) we find \(v = 3.4 \text{ m/s}\). Notice that energy is not conserved in this collision because the initial KE \(= 1/2 (0.5)(5)^2 = 6.25 \text{ J}\), whereas the sum of the final KE \(= 1/2 (0.5)(3)^2 + 1/2 (0.5)(3.4)^2 = 5.1\), amounting to a loss of about 18% of the initial KE.

Example 6.10 Suppose one proton moving with a speed \(v_0\) collides with a second proton initially at rest. If one of the protons emerges at a given angle \(\phi\) from the incident direction, find the speeds of both after the collision and the angle \(\theta\)

(Continued)
at which the second proton emerges from the collision. Work this out in general and then take \( v_0 = 10^6 \) m/s and \( \phi = 30^\circ \).

**Solution:** We are searching for three unknown quantities, and so require three independent equations. We work this problem out without substituting in numbers so that we can learn about the general case. These equations can be obtained from conservation of momentum (two equations, one for the incident direction, say \( x \) and one for the direction perpendicular to that, say \( y \)) and conservation of energy.

Conservation of momentum in the \( x \)-direction gives

\[
P_{0x} = mv_0 = p_{fx} = mv_2 \cos \theta + mv_1 \cos \phi, \tag{1}
\]

where \( m \) is the proton mass, and \( v_1 \) and \( v_2 \) are the protons’ final velocities. Conservation of momentum in the \( y \)-direction gives

\[
P_{0y} = 0 = mv_1 \sin \phi - mv_2 \sin \theta. \tag{2}
\]

Energy conservation gives us the equation

\[
\frac{1}{2} mv_0^2 = \frac{1}{2} mv_1^2 + \frac{1}{2} mv_2^2. \tag{3}
\]

We now have our three equations in three unknowns—this was the physics part of the problem—and the remainder of the problem is to solve for them algebraically. This is a bit complicated, so follow closely. First we can cancel all the \( m \)'s in all three equations and if we then solve for \( v_2 \cos \theta \) and \( v_2 \sin \theta \) in Equations (1) and (2) we have

\[
v_2 \cos \theta = v_0 - v_1 \cos \phi, \tag{4}
\]

and

\[
v_2 \sin \theta = v_1 \sin \phi.
\]

We can then square each of these and add them together to find, using \( \sin^2 \theta + \cos^2 \theta = 1 \), that

\[
v_2^2 = (v_0 - v_1 \cos \phi)^2 + (v_1 \sin \phi)^2,
\]

but from Equation (3), after canceling 1/2 \( m \) from each term, we have that

\[
v_0^2 = v_1^2 + v_2^2 = v_1^2 + (v_0 - v_1 \cos \phi)^2 + (v_1 \sin \phi)^2.
\]

Expanding out the terms in parentheses and combining again we have

\[
v_0^2 = 2 v_1^2 + v_2^2 - 2 v_0 v_1 \cos \phi.
\]

Simplifying this, we have

\[2 v_1 (v_1 - v_0 \cos \phi) = 0,\]

which has the solutions \( v_1 = 0 \) or \( v_1 = v_0 \cos \phi \). The solution \( v_1 = 0 \) gives \( v_2 = \pm v_0 \) indicating a head-on solution in which one proton stops and the other goes on in the forward direction (we must reject the negative solution for \( v_2 \) as unphysical.) The other solution, from Equation (3), gives \( v_2 = \pm v_0 \sin \phi \). In that case to find \( \phi \), after substitution for \( v_1 \) in Equations (4) we have that
In this chapter we have learned how to describe the translational motion of a system of extended objects using the center of mass and momentum conservation. In general such systems will have two other types of motion: overall rotational motion and internal motions. Internal motions include all relative motions of portions of the system other than overall rotational tumbling, including shape changes as well as vibrational motions. We come back to this topic much later in the book in discussions on the structure of matter. Rotational motion is taken up in detail in the next chapter.

\[ v_2 \cos \theta = v_0 - v_0 \cos^2 \phi = v_0 (1 - \cos^2 \phi) = v_0 \sin^2 \phi \]

and

\[ v_2 \sin \theta = v_0 \sin \phi \cos \phi. \]

Dividing these equations we find that

\[ \tan \theta = 1/\tan \phi. \]

Therefore, given an angle \( \theta \) for the first proton, the second emerges such that \( \theta + \phi = 90^\circ \). In our case, if \( v_0 = 10^6 \) and \( \phi = 30^\circ \), we find \( v_1 = 8.7 \times 10^5 \) m/s, \( v_2 = 5.0 \times 10^5 \) m/s and \( \theta = 60^\circ \). You can check these by direct substitution into Equations (1)–(3), after canceling \( m \).

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**CHAPTER SUMMARY**

The momentum of a particle of mass \( m \) is defined as

\[ \mathbf{p} = m \mathbf{v}. \tag{6.1} \]

Using this definition, we can write Newton’s second law for the particle in terms of its momentum as

\[ \mathbf{F}_{\text{net}} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{p}}{\Delta t}. \tag{6.2} \]

If two particles interact in the absence of any external forces, then their total momentum is conserved, meaning that it will remain a constant in time.

A useful concept in discussing collisions is the impulse, defined by the product of the collision force and its duration, and shown to equal the change in momentum of the object:

\[ \text{Impulse} = F \Delta t = \Delta p = p_{\text{final}} - p_{\text{initial}}. \tag{6.5} \]

For a collection of masses \( m_i \), each located at \((x_i, y_i)\), with total mass \( M \), we define the center of mass to be located at the point

\[ x_{\text{cm}} = \sum \left( \frac{m_i}{M} \right) x_i \quad \text{and} \quad y_{\text{cm}} = \sum \left( \frac{m_i}{M} \right) y_i. \tag{6.7} \]

Then for a system of such masses, Newton’s second law can be shown to be

\[ \mathbf{F}_{\text{net, ext}} = \sum \mathbf{F}_{\text{ext}} = M \mathbf{a}_{\text{cm}}. \tag{6.8} \]

or written in term of momentum, as

\[ \mathbf{F}_{\text{net, ext}} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{p}_{\text{cm}}}{\Delta t}. \tag{6.10} \]

In the case of an isolated system, with no external forces acting, the center of mass momentum, equal to the total momentum of the system, is conserved: \( P_{\text{total}} = \) constant. This is a vector equation and, in general, stands for the three independent equations for which each component \((x, y, \text{ and } z)\) of momentum remains constant.
QUESTIONS
1. What are the differences and similarities between momentum and velocity? Between momentum and kinetic energy?
2. Is it possible for the center of mass of a solid object to lie physically outside the object? Give an example or two to support your assertion.
3. For uniform (constant density) objects, is it true that the center of mass must lie along a symmetry axis, if there is one? Give some examples.
4. Explain, in your own words, why only external forces result in a change in the center of mass momentum of a system of interacting particles or an extended object.
5. Carefully define an isolated system. Give some examples and explain why it is that momentum is only conserved for an isolated system.
6. Is a rocket traveling in outer space an example of an isolated system? If so, how can the rocket change its momentum if it is to be conserved?
7. Two identical twins of equal mass are ice skating toward each other at the same speed. What happens when they collide? What happened to their momentum?
8. In a collision of a tennis ball with a racket, why should the tension in the strings of the racket be made as large as possible?
9. When a collision between two objects occurs and there is a net change in the momentum of one object, there are very large forces acting for a very short time. The product of the average force on the object during the collision and the duration of the collision is called the impulse. If a tennis ball of mass m and velocity v bounces off a wall and rebounds with the same speed, what is the impulse on the ball? Why does a new tennis ball bounce higher than an older tennis ball when dropped from the same height?
10. A woman and a man are skating on smooth level ice. Initially, they are in contact and at rest. The man pushes the girl away from him with a force of 30 N. Immediately after they are no longer in contact the girl’s speed is 2 m/s. At the same instant the man’s speed (a) must be zero, (b) must be 2 m/s also, (c) must be 1 m/s, (d) depends on how much force the girl exerts on the man.

MULTIPLE CHOICE QUESTIONS
1. A 3 kg mass has position coordinates (−2, 2 m) and a 1 kg mass has position coordinates (3, 0 m). The center of mass of this system has coordinates (a) (1, 2 m), (b) (−3, 6 m), (c) (−0.75, 1.5 m) (d) (0, 0 m).
2. A 2 kg mass is at x = 0 m, y = +2 m, and a 3 kg mass is at x = 2 m, y = 0 m. The x- and y-coordinates, respectively, of the center of mass of this system are (a) +6/5 m, +4/5 m, (b) +2/5 m, +2/5 m, (c) 0, 0 m, (d) +2 m, +2 m.
3. A 5 kg bowling ball with a center of mass velocity of 4 m/s strikes the padded end of the bowling lane and comes to rest in 0.01 s. The average force exerted on the ball is (a) 400 N, (b) 2000 N, (c) zero, (d) 500 N.
4. The magnitude of the average force on the car during this time is (a) 720 N, (b) 73 N, (c) 420 N, (d) 210 N.
5. The direction of the average force on the car during this time is (a) 38° S of W, (b) 38° N of W, (c) 52° S of W, (d) 52° N of W.

PROBLEMS
1. Find the center of mass of the following sets of point masses.
   (a) A 2 kg mass at x = 5 cm and a 5 kg mass at x = −2 cm
   (b) A 1 kg mass at y = 0 and a 4 kg mass at y = 10 cm
   (c) Three small objects each of the same mass m, located at the following points (0,0), (0,10 cm), (10 cm, 0)
   (d) Point mass m at (0,0), point mass 3m at (0, 5 cm), point mass 5m at (5 cm, 0) and point mass m at (5, 5 cm)
2. Using Table 6.1, find the center of mass of
   (a) A person standing upright with hands at sides
   (b) An outstretched arm and an arm bent upward at the elbow by a right angle
   (c) A person bent over so that there is a right angle between her straight legs and upper body/head and between her upper body and straight arms

Questions 4 and 5 refer to a car weighing 900 N that is heading north at 14 m/s. It makes a sharp turn and heads west at 18 m/s. During the turn, a good luck charm hanging from the rear view mirror is angled from the vertical for a total of 5 s.
3. From the text discussion, you know that the center of mass can be found through “balancing methods”, that is, suspending an object from a point. This procedure indicates that for three equal masses situated at the vertices of an equilateral triangle, the center of mass will be at the intersection of the three angle bisectors of the triangle. From elementary geometry theorems, it is known that the three angle bisector segments intersect at a point that is 2/3 of a segment length away from its angle vertex. Calculate the center of mass for the mass arrangement shown and compare its position to the intersection of the angle bisectors. Note that because the height of the triangle is $a = \frac{3}{2}$, the method is a physical manifestation of the theorem that the bisectors of angles of an equilateral triangle intersect at the center of mass of the triangle (usually called the “centroid” by mathematicians). This is true whether the physical triangle is constructed of sides only, of similar and uniform cross-section, or if the triangle is a uniform plate.

4. Calculate the center of mass for three equal masses situated at the vertices of a 3-4-5 right triangle.

5. Calculate the center of mass for the arrangement of three masses also situated at the vertices of a 3-4-5 right triangle, but where the masses are in the ratio 3:4:5, with the largest opposite the hypotenuse and the smallest opposite the shortest side. Compare the result with the previous problem.

6. By symmetry, the center of mass of a uniform regular polygon is at its center. Similarly, this is true for an arrangement of equal masses situated at the vertices of such a polygon. Where is the center of mass for each of the arrangements shown, where only a subset of the vertices of the polygon is occupied by masses? Make sure you arrange the coordinate framework to take advantage of any remaining symmetry. (Hint: Missing masses can be represented as $M - M = 0$ mass, so that the sum of a negative mass can be added to the situation with a full complement of masses at the vertices.)

7. Consider the three spherical masses shown. How far to the right of $m_2$ should $m_3$ be so that the center of mass of the entire arrangement is located exactly at the position of $m_2$?

8. Consider a uniform linear arrangement of ten masses, that is, with equal spacing between, ranging from 1 to 10 kg, each 1 kg more than the previous. Where is the center of mass of the assemblage?

9. Three uniform spheres of radii $R$, $2R$, and $3R$ lie in contact with each other from left to right in the order given with their centers along the $x$-axis. Remembering that the volume of a sphere is given by $(4/3)\pi r^3$, find the position of the center of mass of the three spheres as measured from the left edge of the smallest sphere.

10. Find the center of mass of a screwdriver with the following characteristics: a wooden cylindrical handle (density of wood $= 0.5 \times 10^3$ kg/m$^3$; cylinder length and diameter $= 10$ and 2 cm) and a steel cylindrical rod (density of steel $= 7.8 \times 10^3$ kg/m$^3$; 15 cm long and 0.5 cm in diameter, with an additional 3 cm flat uniformly tapered head with a triangular cross-section).

11. Three uniform rods (identical except for their lengths) form the right triangle shown with coordinates measured in meters.
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14. Tennis pros can often serve the ball at speeds in
13. A 0.1 kg ball bounces perpendicularly off a wall with
12. A ball of 0.5 kg mass is dropped from rest at a

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(a) Write an equation that governs the momentum of the ball and pendulum arm during the collision and solve this for the initial velocity of the ball.
(b) After the collision, mechanical energy is conserved. Write an equation that shows conservation of mechanical energy immediately after the collision to the point where the pendulum arm and ball come to rest momentarily at the angle \( \theta \). Solve this equation for the velocity of the ball and pendulum arm after the collision. Express your answer in terms of \( R_{cm} \) and \( \theta \) and you may ignore any rotational motion of the arm.
(c) Using the equations that you have written in parts (a) and (b) what is the expression for and the value of the initial velocity of the ball?
(d) What fraction of the initial kinetic energy of the ball has been lost in the collision?

22. An automobile has a mass of 2300 kg and a velocity of 16.0 m/s. It makes a rear-end collision with a stationary car whose mass is 1800 kg. The cars lock bumpers and skid off together with their wheels locked.
(a) What is the velocity of the center of mass of the two-car system?
(b) What is the velocity of the two cars just after the collision?
(c) What is the change in total kinetic energy during the collision?
(d) What is the magnitude of the impulse experienced by the 2300 kg car?
(e) If the duration of the collision is 0.100 s, what is the magnitude of the average force experienced by the 2300 kg car?
(f) What is the magnitude of the average force experienced by the 1800 kg car?

23. A 0.01 kg bullet traveling at 300 m/s ricochets off a stationary steel block of 2 kg mass. The bullet is deflected by 5° and travels at 250 m/s after the collision. Find the velocity (magnitude and direction) of the block after the collision.

24. A 10 g projectile is fired at 500 m/s into a 1 kg block sitting on a frictionless surface. The projectile lodges in the center of the block, and both move off together.
(a) What is the final velocity of the block after the collision?
(b) The block slides along the frictionless surface some distance and then encounters a ramp, which slopes up at an angle of 60°. What distance does the block travel along the surface of the ramp before coming to a stop?
(c) If the coefficient of friction between the block and the ramp is \( \mu_k = 0.2 \), how far does the block slide up the ramp before stopping?

25. A proton moving with an initial velocity \( v_{ix} \) in the \( x \)-direction, as shown in the figure, collides elastically with another proton that is initially at rest. If the two protons have equal speeds after the collision, what is the speed of each proton after the collision in terms of \( v_{ix} \), and what are the directions of the velocity vector after the collision?