Electromagnetics from a quasistatic perspective

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Quasistatic models provide intermediate levels of electromagnetic theory in between statics and the full set of Maxwell's equations. Quasistatics is easier than general electrodynamics and in some ways more similar to statics, but exhibits more interesting physics and more important applications than statics. Quasistatics is frequently used in electromagnetic modeling, and the pedagogical potential of electromagnetic simulations gives additional support for the importance of quasistatics. Quasistatics is introduced in a way that fits into the standard textbook presentations of electrodynamics.
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I. INTRODUCTION

Applications of electrodynamics may be in the static, quasistatic, or high frequency regime. Quasistatics is neglected in most textbooks and the purpose of this paper is to fill this gap in a course on the level of Griffiths' textbook.[1] The reasons to do so include the following:

(1) Quasistatics is useful for a better understanding of the transition from statics to dynamics. Consider, for example, the question of when are the laws of Coulomb and Biot-Savart applicable outside the static domain? This question and similar ones are considered in many papers, and there has been much confusion (see the discussion in Ref. 2 and references therein). Another confusing issue concerns the appearance of time-varying electric fields in regions where the magnetic field is absent (such as outside a long solenoid or toroidal coil with a time-varying current).[3–7] Still another example is the standard textbook motivation for the displacement current where it is introduced to make the equations of electrodynamics consistent with charge conservation. The physical impact of this term includes electromagnetic waves in free space. Why should waves in free space without charges and currents follow from charge conservation?[8] We will see that a quasistatic perspective allows us to clarify these confusing issues.
(2) An analysis of the full set of Maxwell's equations is frequently difficult and is not necessary in quasistatic situations.[9]
(3) The numerical solution of the time-dependent hyperbolic Maxwell equations in three dimensions can be replaced by elliptic (or parabolic) equations that can be solved in simpler ways.[10–12]
(4) Quasistatics has much in common with statics. In statics we may formulate the theory either in terms of the laws of Coulomb, Biot-Savart, and charge conservation, or equivalently as the static limit of Maxwell's equations. Quasistatics has a similar structure with two alternative and equivalent formulations.

This paper is organized as follows. The basic theory of quasistatics is presented in Sec. II. Section II A formulates the quasistatic models in terms of integral expressions like the laws of Coulomb and Biot-Savart. In Sec. II B the same models are formulated in terms of differential equations. An important feature of quasistatics is instantaneous interaction at a distance. The corresponding $c \to \infty$ limit of electrodynamics is used to derive quasistatic theories in Sec. II C. Sections II D–II F discuss the quasistatic theory for the electromagnetic potentials, the fields from a moving point charge, and the quasistatic Poynting theorems. Alternatives to the textbook derivation of Maxwell's equations are discussed in Sec. III from a quasistatic perspective. Section IV discuss the laws of Coulomb and Biot-Savart outside the static domain. Necessary and sufficient and conditions for the validity of these laws are given. The appearance of time-varying electric fields in regions where the magnetic field is absent is discussed in Sec. V. In Sec. VI we consider the possibility of doing simulations in courses using the quasistatic equations for eddy currents as an example. A summary is given in Sec. VII.

II. QUASISTATICS

Quasistatics in electrodynamics refers to a regime where "the system is small compared with the electromagnetic wavelength associated with the dominant time scale of the problem."[13] The fields are propagated instantaneously so we are dealing with a $c \to \infty$ limit. We consider in this paper three major quasistatic models, including the EQS (electroquasistatics),[9] MQS (magnetoquasistatics),[9] and Darwin models.[10–12] EQS includes capacitive but not inductive effects, MQS includes inductive but not capacitive effects, and the Darwin model includes both capacitive and inductive effects. The Biot-Savart law is valid in all three models, and Coulomb's law is valid only in EQS. In the MQS and Darwin models there is an additional contribution to the electric field due to the $\partial \mathbf{B}/\partial t$ term in Faraday's law.[14]

A. From the laws of Coulomb, Biot-Savart, and Faraday to the EQS, MQS, and Darwin models

Dynamical systems that proceed from one state to another as though they are static (at each fixed time) are said to be quasistatic. For electromagnetism the static theory builds on the Coulomb and Biot-Savart laws together with the static continuity equation. Quasistatics is obtained by allowing for time dependence in the otherwise time independent laws

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0} \int\int\int \frac{\rho(\mathbf{r}',t)\hat{\mathbf{R}}}{R^2}\, d\tau' \tag{1}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int\int\int \frac{\mathbf{J}(\mathbf{r}',t) \times \hat{\mathbf{R}}}{R^2}\, d\tau', \qquad (2)$$

where $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, $R=|\mathbf{R}|$, and $\hat{\mathbf{R}}=\mathbf{R}/R$. It would be strange to keep the static continuity equation unchanged in this time-dependent situation, so we replace it by the usual continuity equation[15]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \qquad (3)$$

Equations (1)–(3) constitute the EQS model, which includes capacitive but not inductive effects. Charge may be accumulated in this model but this accumulation requires work; energy may as usual be associated with the electric field. Magnetic energy is outside the scope of EQS because there is no magnetically induced electric field and accordingly no back emf. No work is required to create a magnetic field by starting an electric current.

We now improve on the EQS model by including electromagnetic induction (and thereby magnetic energy). Then $\partial \mathbf{B}/\partial t$ also acts as a source of electric fields and the total electric field becomes the sum of two parts,

$$\mathbf{E} = \mathbf{E}_C + \mathbf{E}_F, \qquad (4)$$

where E_C is the Coulomb electric field,

$$\mathbf{E}_C(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0} \int\int\int \frac{\rho(\mathbf{r}',t)\hat{\mathbf{R}}}{R^2}\, d\tau', \qquad (5)$$

and E_F is the Faraday electric field, which may be defined by a Biot-Savart-like integral expression as [cf. Eq. (2)][16]

$$\mathbf{E}_F(\mathbf{r},t) = -\frac{1}{4\pi} \int\int\int \frac{\partial \mathbf{B}(\mathbf{r}',t)}{\partial t} \times \frac{\hat{\mathbf{R}}}{R^2}\, d\tau'. \qquad (6)$$

We note that $\mathbf{E}_F$ satisfies the equations

$$\nabla \times \mathbf{E}_F = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{E}_F = 0, \qquad (7)$$

and the corresponding equations for $\mathbf{E}_C$ are

$$\nabla \times \mathbf{E}_C = 0, \quad \nabla \cdot \mathbf{E}_C = \frac{\rho}{\varepsilon_0}. \qquad (8)$$

Equations (5) and (6) are the unique solutions of Eqs. (7) and (8), provided appropriate boundary conditions at infinity are used.[17]

The Darwin model may be defined by Eqs. (2)–(6) and includes both capacitive and inductive phenomena. Note that for given current and charge densities we directly obtain the electromagnetic fields in terms of integrals without the appearance of time retardation. Instead the integral expressions of the laws of Coulomb and Biot-Savart play an important role in this model.

The MQS model is obtained from the Darwin model if the usual continuity equation is replaced by the static equation

$$\nabla \cdot \mathbf{J} = 0. \qquad (9)$$

Thus the MQS model may be defined by Eqs. (2), (4)–(6), and (9). It differs from both EQS and Darwin because it includes inductive but not capacitive effects. A confusing feature of MQS is that the continuity equation may be violated by MQS solutions. Only stationary currents are allowed in MQS and these currents cannot explain changes in the charge density. Thus we cannot interpret the currents in MQS in terms of charge transport. Ampère's law, which implies Eq. (9), is valid in the MQS model, but not for the EQS or Darwin models (see Sec. II B). Thus Ampère's law is not always valid in quasistatics. Griffiths and Heald remark that "The application of Ampère's law in quasistatic situations can be an extremely delicate matter."[2]

The textbook by Haus and Melcher[9] develops its understanding of electrodynamics by using both EQS and MQS. This development is of particular significance in the relation between electromagnetic field theory and circuit theory. EQS involves capacitance features and MQS inductance features. For systems involving capacitance and inductance both models are needed. This need is a complicating feature if we wish to model the entire system numerically because we would have to divide the system into EQS and MQS subregions with appropriate continuity conditions at the interfaces. A better alternative may be to use the Darwin model, which embraces all the physics contained in the EQS and MQS models. In Sec. VI B we discuss eddy currents in an inhomogenous conductor and use the Darwin model because both capacitance (due to charge accumulation at surfaces and other inhomogeneities) and inductance appear.

B. From Maxwell's equations to EQS, MQS, and Darwin

We now consider the formulation of quasistatics in terms of differential equations. The starting point is Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \qquad (10)$$

$$\nabla \cdot \mathbf{B} = 0, \qquad (11)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad (12)$$

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right). \qquad (13)$$

In the presence of polarizable/magnetizable media, it is convenient to rewrite Maxwell's equations by introducing the D and the H fields. For simplicity, only E and B fields will be used here, but it is straightforward to modify the expressions so that a formulation with D and H fields is obtained.

We now formulate the EQS, MQS, and Darwin models as approximations of Maxwell's equations (10)–(13). The EQS model is obtained from Maxwell's equations by neglecting $\partial \mathbf{B}/\partial t$ in Faraday's law, Eq. (12),

$$\nabla \times \mathbf{E} = 0. \qquad (14)$$

MQS is obtained by neglecting $\partial \mathbf{E}/\partial t$ in the Ampère-Maxwell law, Eq. (13), resulting in the usual Ampère law[18]

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \qquad (15)$$

To obtain the Darwin model we do not neglect all of $\partial \mathbf{E}/\partial t$, but keep the Coulomb part of the E field [defined by Eq. (5)] and replace Eq. (13) by the Ampère-Darwin equation[19]

Table I. The two equivalent definitions of the quasistatic models EQS, MQS, and Darwin.

Model	Biot-Savart-Coulomb	Maxwell
EQS	Eqs. (1)–(3)	Eqs. (10), (11), (13), and (14)
MQS	Eqs. (2), (4)–(6), and (9)	Eqs. (10)–(12) and (15)
Darwin	Eqs. (2)–(6)	Eqs. (10)–(12), (16), and (8)

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}_C}{\partial t}\right). \tag{16}$$

Note an important difference between Eqs. (15) and (16). Equation (15) implies Eq. (9) and may violate charge conservation in dynamic situations. The Ampère-Darwin equation, like the Ampère-Maxwell equation (13), is consistent with the continuity equation (3). Also note that the term in the parentheses of Eq. (16) is the transverse (that is, divergenceless) part of the current density.

We see that the quasistatic models may be defined by focusing on the laws of Coulomb and Biot-Savart or as approximations to Maxwell's equations. These two formulations are summarized in Table I. The proof that the corresponding columns define the same models involves only standard procedures.[1] In electrostatics we start from Coulomb's law and obtain the divergence and curl of the electric field. By Helmholtz's theorem we have the equivalence (assuming appropriate conditions at infinity[17])

$$\text{Eq. (1)} \Leftrightarrow \text{Eqs. (10) and (14)}. \tag{17}$$

In the same way we have that Eq. (5) is equivalent to Eq. (8). In magnetostatics we start from the law of Biot-Savart and obtain the divergence and curl of the magnetic field. We find the equivalence

$$\text{Eqs. (2) and (9)} \Leftrightarrow \text{Eqs. (11) and (15)}. \tag{18}$$

The relations (17) and (18) remain valid if we allow for time dependence where the time appears only as a parameter. However, in a time dependent situation it is logical to use the general continuity equation (3) instead of the static one (9). Then instead of Eq. (18) we find

$$\text{Eqs. (2) and (3)} \Leftrightarrow \text{Eqs. (11) and (16)}. \tag{19}$$

Thus the Ampère-Darwin equation is an implication of Biot-Savart's law combined with the continuity equation. The relation (19) is not included in most textbooks, but has (more or less explicitly) appeared in Refs. 20–22.

Equation (6) for the Faraday electric field is formally analogous to the Biot-Savart law (2) for the magnetic field. Formally similar to Eq. (18) is the equivalence

$$\text{Eqs. (6) and (11)} \Leftrightarrow \text{Eqs. (7) and (11)}. \tag{20}$$

Table II summarizes some of the laws of electromagnetics and their validity in the EQS, MQS, and Darwin models. The equations are familiar from standard electromagnetics with the exception of the Ampère-Darwin equation (16).

C. A limit of instantaneous propagation in Maxwell's equations

In Sec. II A we obtained the Darwin model from EQS by including Faraday's law in a straightforward way. An alternative procedure is to follow the textbook derivation of Maxwell's equations,[1] but when the displacement current is added to Ampère's law for consistency with the continuity equation, we find the Ampère-Darwin equation (16) rather than the Ampère-Maxwell equation (13). These derivations of Darwin's model might seem arbitrary, and it would be desirable to obtain Darwin's model from Maxwell's equations by using a more systematic method.

Table II. Validity of several familiar laws in the EQS, MQS, and Darwin models.

Equation No.	Description	EQS	MQS	Darwin
(1)	Coulomb's law	yes	no	no
(2)	Biot-Savart's law	yes	yes	yes
(3)	Continuity equation	yes	no	yes
(9)	$\nabla \cdot \mathbf{J}=0$	no	yes	no
(10)	Gauss law	yes	yes	yes
(11)	$\nabla \cdot \mathbf{B}=0$	yes	yes	yes
(12)	Faraday's law	no	yes	yes
(13)	Ampère-Maxwell	yes	no	no
(14)	$\nabla \times \mathbf{E}=0$	yes	no	no
(15)	Ampère's law	no	yes	no
(16)	-Darwin	yes	no	yes

A feature of quasistatic approximations to Maxwell's equations is the instantaneous propagation of fields. Thus it should be possible to consider quasistatics as the limit $c \to \infty$ of Maxwell's equations. This limit is singular and the procedure must be further specified. We take the absence of time retardation as being the most salient property of quasistatics. Let us express Maxwell's equations in terms of the potentials $(V, \mathbf{A})$ so that the retarded time appears explicitly. We use the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0, \tag{21}$$

and follow the derivation in Nielson and Lewis[10] (the reason for not using the Lorentz gauge will be considered soon). The equations for the potentials become (see Ref. 1, p. 421)

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}, \tag{22}$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \nabla \frac{\partial V}{\partial t}. \tag{23}$$

Time retardation appears explicitly if we solve Eq. (23) for the vector potential in terms of an integral in the usual way. The omission of retarded time in this integral is the same as excluding the second order time derivative so that

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \nabla \frac{\partial V}{\partial t}. \tag{24}$$

The Darwin model in terms of potentials consists of Eqs. (21), (22), and (24).[23] The usual Darwin model given by Eqs. (10)–(12), (16), and (8) is obtained by using

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E}_C = -\nabla V. \tag{25}$$

Why didn't we use the Lorentz gauge? It is straightforward to make the corresponding calculations also in this gauge. We then obtain a model approximating Maxwell's equations that is close to the Darwin model with the Ampère-

Darwin law as one of the equations. Gauss' law (10) is not obtained, but a new term appears in the corresponding equation. We prefer not to change Gauss' law and this problem does not appear if we use the Coulomb gauge in the approximation procedure. Using the Coulomb gauge also has the desirable consequence that the Coulomb part of the E field is given by the scalar potential $\mathbf{E}_C = -\nabla V$ and the Faraday part by the vector potential $\mathbf{E}_F = -\partial \mathbf{A}/\partial t$.

Note that although Maxwell's equations are gauge invariant, we would not expect an approximation to be so. Thus, the approximations in the Coulomb and Lorentz gauges are different.

D. The potential representations of EQS, MQS, and Darwin

The magnetic field $\mathbf{B}$ is written as $\nabla \times \mathbf{A}$ in all models. For MQS the vector potential satisfies

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \tag{26}$$

or in integral form

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int\int\int \frac{\mathbf{J}}{R} \, d\tau'. \tag{27}$$

For EQS and Darwin the corresponding equation is Eq. (24) or

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int\int\int \frac{(\mathbf{J} + (\mathbf{J} \cdot \hat{\mathbf{R}})\hat{\mathbf{R}})}{2R} \, d\tau'. \tag{28}$$

It takes some manipulations involving partial integration and the continuity equation to derive Eq. (28) from Eq. (24). The scalar potential is just the Coulomb potential for EQS, MQS, and Darwin,

$$V = \frac{1}{4\pi\varepsilon_0} \int\int\int \frac{\rho}{R} \, d\tau'. \tag{29}$$

The electric field for EQS is the Coulomb field

$$\mathbf{E} = -\nabla V, \tag{30}$$

and the electric field for MQS and Darwin also includes the magnetically induced electric field so that

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \tag{31}$$

E. Fields of a moving point charge

We now consider the quasistatic fields from a point charge in motion. Coulomb's law is often taken as a starting point for electrostatics. It is conceptually simple to discuss interactions between point charges in this way, so why not develop all of electrodynamics by generalizing this approach? This approach is discussed by Griffiths.[1] Two major problems are that the force between two point charges depends not only on their separation, but also on their velocities and accelerations. Furthermore, it is the position, velocity, and acceleration at the retarded time of the other particle that enters.

The second problem is far more challenging. However, time retardation vanishes in the quasistatic limit, and the approach with interacting point charges becomes instructive. In the point charge approach to electrodynamic models we automatically include the continuity equation (3). Therefore this section concerns EQS and Darwin, but not MQS.

The fields from a point charge Q in EQS are

$$\mathbf{E}(\mathbf{r},t) = \frac{Q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{R}}}{R^2}, \tag{32}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 Q}{4\pi} \frac{\mathbf{v} \times \hat{\mathbf{R}}}{R^2}. \tag{33}$$

The position of Q is $\mathbf{r}'(t)$ and the velocity is $\mathbf{v} = d\mathbf{r}'/dt$. From Eqs. (32) and (33) we obtain Eqs. (1) and (2), and the continuity equation (3) is implied by the point charge description.

Let us now consider the Darwin model. Equation (33) remains valid and Eq. (32) gives only the Coulomb part of the electric field

$$\mathbf{E}_C(\mathbf{r},t) = \frac{Q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{R}}}{R^2}. \tag{34}$$

The magnetically induced part $\mathbf{E}_F$ must also be included. It is convenient to start over again using the electromagnetic potentials. The potentials are

$$V(\mathbf{r},t) = \frac{Q}{4\pi\varepsilon_0 R}, \tag{35}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 Q}{4\pi} \frac{\mathbf{v} + (\mathbf{v} \cdot \hat{\mathbf{R}})\hat{R}}{2R}. \tag{36}$$

The scalar potential (35) is the Coulomb potential and the vector potential (36) may be found from Eq. (28). An alternative derivation is by the ansatz

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 Q}{4\pi} \frac{\mathbf{v}}{R} + \nabla \phi, \tag{37}$$

where the scalar field ϕ is determined by the Coulomb gauge condition (21). It takes a little algebra to find

$$\phi = -\frac{\mu_0 Q}{8\pi} \mathbf{v} \cdot \hat{\mathbf{R}}, \tag{38}$$

and thus Eq. (36) follows.[24] The electromagnetic fields are related to the potentials in the usual way, Eq. (25). The magnetically induced part of the electric field is

$$\mathbf{E}_F = -\partial \mathbf{A}/\partial t. \tag{39}$$

Equation (39) involves the acceleration $\mathbf{a} = d\mathbf{v}/dt$ of Q. The Darwin model should thus follow from the potentials (35) and (36) of a point charge, and it is sufficient to check Eqs. (2)–(6). Only Eq. (6) is not obvious and the easiest approach is to use the equivalent differential equation (7). These equations are trivially satisfied by Eq. (39) using the Coulomb gauge and $\mathbf{B} = \nabla \times \mathbf{A}$.

The usual textbook approach to electrostatics begins with Coulomb's law (32) for a charge at rest. Sometimes the corresponding approach to magnetostatics takes Eq. (33) as a starting point but with a moving charge. This procedure might seem incorrect because the system is time dependent. Most textbooks take this objection seriously and instead start from the Biot-Savart law (2) with a stationary current. The expression for the Biot-Savart law is complicated, and it is a relief that the magnetostatic differential equations (11) and (15) are formally similar to and not much more difficult than the electrostatic equations (10) and (14). Quasistatics is easier than statics in this respect because it allows for the use of Eq. (33).

A conventional Lagrangian description of the electromagnetic interaction between two or more charged particles is possible only if time retardation is neglected. It is straightforward to give such a formulation constructed from the potentials (35) and (36).[25] This construction results in the Darwin Lagrangian, which was first obtained by Oliver Heaviside in 1891.[26,27] It was found again by Darwin in 1920 by an expansion of the Lienard-Wiechert potentials.[28] It turns out that the terms up to order $(v/c)^2$ inclusive are included in the Darwin Lagrangian (see Ref. 13, p. 596).

F. The Poynting theorem

We follow the standard derivation of Poynting's theorem using the quasistatic models. That is, we add the two equations obtained by taking the scalar product of the curl E equation with the B field and the scalar product of the curl B equation with the E field. After the usual manipulations we obtain for EQS

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right) = -\mathbf{E}\cdot\mathbf{J}. \tag{40}$$

For MQS we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2\mu_0}B^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right) = -\mathbf{E}\cdot\mathbf{J}, \tag{41}$$

and for the Darwin model we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E_C^2 + \frac{1}{2\mu_0}B^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} + \varepsilon_0\frac{\partial V}{\partial t}\frac{\partial \mathbf{A}}{\partial t}\right) = -\mathbf{E}\cdot\mathbf{J}. \tag{42}$$

In EQS only the electric field is associated with the energy. Building up a magnetic field costs no energy in this model due to the absence of a counteracting induced E field. In MQS only the magnetic field has energy. Changes in the electric field are associated with changes in the charge density. The continuity equation is not satisfied, and there is no charge transport for which we could calculate the required energy. The Darwin model includes both magnetic and electric energy. Note that only the Coulomb part of the E field is associated with energy. An important qualitative difference between EQS and MQS on the one hand and Darwin's model on the other is the possibility of natural resonances in the latter. Because Darwin includes both capacitive and inductive features, we may in principle use these equations to model, for example, a field theoretic manifestation of a LC circuit.

III. IF MAXWELL HAD WORKED IN BETWEEN AMPÈRE AND FARADAY

We consider in this section various procedures to find the full set of Maxwell equations starting from the laws of Coulomb and Biot-Savart. The quasistatic perspective is useful and the EQS, MQS, and Darwin models will appear as intermediate stages. The title of this section refers to the possibility of introducing the displacement current before Faraday's law.[29]

Let us start with the standard textbook procedure of finding Maxwell's equations.[1] From the static laws of Coulomb and Biot-Savart we find, by using the static continuity equation, the static limit of Maxwell's equations. By introducing magnetic induction, we then obtain MQS, that is, "Electrodynamics before Maxwell" as in Ref. 1. The final step, motivated by the need for consistency with charge conservation (3), is to introduce the displacement current. At this point a question mentioned in Sec. I may arise: Maxwell's equations include waves in free space but why should they follow from charge conservation? Quasistatics is useful for discussing this issue. It is not necessary to introduce all of the displacement current to save charge conservation. It is sufficient to include the Coulomb part of it,[8] and obtain the Darwin model. At this point no new physics appears. The step from Darwin to Maxwell's equations may be motivated by the symmetry and beauty of the equations. The explicit appearance of the Coulomb electric field in the Darwin model is an unsatisfactory feature, and it is easily fixed by replacing it by the total electric field. Amazing new physics now appears and is accompanied by beautiful new mathematical structures such as the Lorentz invariance of both the Maxwell equations and the trajectories of test charges.

Jammer and Stachel discuss what might have happened if Maxwell had worked in between Ampère and Faraday.[29] As in the usual textbook derivation of Maxwell's equations, we may first derive the static limit of Maxwell's equations from the laws of Coulomb and Biot-Savart combined with the static continuity equation. At this stage Maxwell might have added the displacement current before the discovery of Faraday's law. This addition would result in the EQS model, which is Galilean invariant.[29] It is suggested that the discovery of Faraday's law "would have confronted physicists with the dilemma: give up the Galilean relativity principle for electromagnetism or modify it (special relativity). One may imagine a new pedagogical approach to electromagnetic theory, in which the displacement current and the Galilean relativity principle are introduced before the induction term is discussed." This approach is less striking than it first seems. We would like to include the appropriate invariance structure not only for the field equations but also for the trajectories of test charges. Here we have a problem in EQS. Only the electric part of the Lorentz force is Galilean invariant, so we would have to neglect the magnetic force on the test charge. The problem of finding satisfactory quasistatic and Galilean invariant approximations to Maxwell's equations is not an easy one.[30]

A third approach to motivating Maxwell's equations avoids taking the static limit of Maxwell's equations by allowing for a (trivial) time dependence in the laws of Coulomb, Eq. (1), and Biot-Savart, Eq. (2). We obtain the differential equations of EQS, Eqs. (10), (11), (13), and (14), if we also use the (time-dependent) continuity equation (3). Maxwell's equations now follow directly when Faraday's law is introduced. In this case the question of why the magnetic induction experiments of Faraday imply electromagnetic waves and time retardation appears. The answer is somewhat hidden: The electric field in EQS is a pure Coulomb field $\mathbf{E}_C$, but when we include magnetic induction, we also obtain the Faraday electric field $\mathbf{E}_F$. It is now not clear whether we should include $\mathbf{E}$ or $\mathbf{E}_C$ in Eq. (13). In the latter case we obtain the Darwin model and no qualitatively new physics appears (only what is needed to explain the experiments of Faraday). The step from the Ampère-Darwin to the Ampère-Maxwell law has already been discussed.

IV. THE LAWS OF COULOMB AND BIOT-SAVART IN TIME-DEPENDENT THEORY

The laws of Coulomb and Biot-Savart provide a starting point for the static theory of electromagnetics. How should the static laws of Coulomb and Biot-Savart be generalized to time-dependent theory? Consider the trivial generalization given by Eqs. (1) and (2) where time is included as a parameter. What is the significance of these formulas? The first question was considered by Jefimenko and the resulting formulas involve integrals involving the retarded time.[31] Jefimenko's generalized laws of Coulomb and Biot-Savart have been used by Griffiths and Heald to address the second question. They find the following.[2]

(a) The generalized Biot-Savart law reduces to the standard one if $\partial^2\mathbf{J}/\partial t^2=0$. (From the continuity equation they also find $\partial^3\rho/\partial t^3=0$.)
(b) The generalized Coulomb law reduces to the standard one if $\partial\mathbf{J}/\partial t=0$ (and thus also $\partial^2\rho/\partial t^2=0$).
(c) The law of Ampère holds if $\partial^2\mathbf{J}/\partial t^2=0$ and $\partial\rho/\partial t=0$.

These results are all sufficient conditions for the Biot-Savart, Coulomb, and Ampère laws, but only (c) also provides necessary conditions. Necessary and sufficient conditions will be formulated in the following.

It is natural to consider the validity of Coulomb and Biot-Savart in time-dependent theory from the perspective of quasistatics. Statics is just a particular case of the general Maxwell equations but quasistatics is an approximation. When is a solution in quasistatics an exact solution to Maxwell's equations? We will show that a solution of the Darwin model also solves Maxwell's equations if and only if the current has the form

$$\mathbf{J}(\mathbf{r},t)=\mathbf{a}(\mathbf{r})t+\mathbf{b}(\mathbf{r})-\varepsilon_0\frac{\partial\mathbf{E}_C(\mathbf{r},t)}{\partial t}, \tag{43}$$

where the Coulomb field is defined by Eq. (5) and the vector fields $\mathbf{a}$ and $\mathbf{b}$ satisfy $\nabla\cdot\mathbf{a}=\nabla\cdot\mathbf{b}=0$. Formally we write this statement as

$$\text{Maxwell}+\text{Biot-Savart}\Leftrightarrow\text{Darwin}+\text{Eq. (43)}. \tag{44}$$

The last term in Eq. (43) may appear new and unfamiliar in the context of the true current. (This term with the opposite sign is familiar as a part of the displacement current.) This term is the longitudinal (that is, curlless) part of the current. Thus consider any current density $\mathbf{J}$ and express it in accordance with the Helmholtz theorem as $\mathbf{J}=\mathbf{J}_T+\mathbf{J}_L$, where $\mathbf{J}_T$ is transverse (divergenceless) and $\mathbf{J}_L$ is longitudinal.[17,32] It follows from the continuity equation and the time derivative of Eq. (8) that $\mathbf{J}_L=-\varepsilon_0\partial\mathbf{E}_C/\partial t$.

Let us compare Eq. (44) with statement (a). The latter may be reformulated in a way similar to Eq. (44), but then we only obtain the left ($\Leftarrow$) implication and the last term in Eq. (43) is then assumed to be linear in time. Result (a) is thus restricted to the case of currents with a linear time dependence and constitutes a special case of the general result Eq. (44). The exact solution to Maxwell's and Darwin's models with a nonlinear time dependence is given in the following examples.

We now derive Eq. (44) and start with the right ($\Rightarrow$) implication. That Darwin is satisfied follows directly from the equivalence (19). Then both the laws of Ampère-Maxwell and Ampère-Darwin are satisfied, implying that $\partial\mathbf{E}_F/\partial t=0$. From Eq. (7) it follows that $\partial^2\mathbf{B}/\partial t^2=0$. From the second time derivative of Ampère-Darwin and Eq. (11) we now obtain

$$\frac{\partial^2}{\partial t^2}\left(\mathbf{J}+\varepsilon_0\frac{\partial\mathbf{E}_C}{\partial t}\right)=0, \tag{45}$$

and Eq. (43) easily follows, including the conditions $\nabla\cdot\mathbf{a}=\nabla\cdot\mathbf{b}=0$. Consider next the left implication in Eq. (44). By substituting Eq. (43) into the Maxwell-Darwin equation, we find $\nabla\times\mathbf{B}=\mu_0(\mathbf{a}t+\mathbf{b})$, so that $\mathbf{B}$ is at most linear in t. From the time derivative of Eq. (7) we then find $\partial\mathbf{E}_F/\partial t=0$. In this case the Ampère-Darwin and the Ampère-Maxwell laws are the same and the solution to the Darwin model also solves Maxwell's equations.

We have found a necessary and sufficient condition (44) for the validity of the law of Biot-Savart within Maxwell's equations. Thereby the result (a) of Griffiths and Heald has been generalized. Let us now consider the corresponding result for Coulomb's law and allow for a current of the form

$$\mathbf{J}(\mathbf{r},t)=\mathbf{b}(\mathbf{r})-\varepsilon_0\frac{\partial\mathbf{E}_C(\mathbf{r},t)}{\partial t}, \tag{46}$$

where $\nabla\cdot\mathbf{b}=0$. We will show that

$$\text{Maxwell}+\text{Coulomb}\Leftrightarrow\text{EQS}+\text{Eq. (46)}. \tag{47}$$

We start with the right implication in Eq. (47). From Coulomb's law it follows that the E field is conservative so that EQS is obtained from Maxwell. From Faraday's law it follows that $\partial\mathbf{B}/\partial t=0$ and from the time derivative of Ampère-Maxwell we find (using $\mathbf{E}=\mathbf{E}_C$)

$$\frac{\partial}{\partial t}\left(\mathbf{J}+\varepsilon_0\frac{\partial\mathbf{E}_C}{\partial t}\right)=0, \tag{48}$$

and Eq. (46) easily follows. To obtain the left implication of Eq. (47), we first substitute Eq. (46) into the Ampère-Maxwell equation and obtain $\nabla\times\mathbf{B}=\mu_0\mathbf{b}$. This relation together with Eq. (11) results in a time-independent B field and thus Faraday's law is satisfied. The proof of Eq. (47) is now complete and the result (b) is generalized.

Let us finally consider the result (c) involving Ampère's law. This condition turns out to be not only sufficient but also necessary. Consider a current density of the form

$$\mathbf{J}(\mathbf{r},t)=\mathbf{a}(\mathbf{r})t+\mathbf{b}(\mathbf{r}), \tag{49}$$

where the vector fields $\mathbf{a}$ and $\mathbf{b}$ satisfy $\nabla\cdot\mathbf{a}=\nabla\cdot\mathbf{b}=0$. Then

$$\text{Maxwell}+\text{Ampère}\Leftrightarrow\text{MQS}+\text{Eq. (49)}+\text{Eq. (3)}. \tag{50}$$

Consider first the right implication. Both Ampère's law and Ampère-Maxwell are valid so $\partial\mathbf{E}/\partial t=0$. By using the time derivative of Faraday's law, we find $\partial^2\mathbf{B}/\partial t^2=0$, so from the second time derivative of Ampère's law we have $\partial^2\mathbf{J}/\partial t^2=0$. Equation (49) follows if we also use the fact that the current has zero divergence. The left implication in Eq. (50) is obtained by reversing this procedure.

We now consider several examples involving the results we have found.

Example 1. Consider an (infinitely) long solenoid or a toroidal coil with time-varying current $I(t)$. We assume that the coils are wound so that we may neglect the axial current in the solenoid and the poloidal current in the toroid. What are the electromagnetic fields? Within the quasistatic approximation we obtain (exactly) zero B field outside the coils (from the Biot-Savart law and symmetry). The MQS and Darwin models coincide because there is no charge density and accordingly $\mathbf{E}_C=0$. There is a magnetically induced electric field $\mathbf{E}=\mathbf{E}_F$ outside the coils in spite of the fact that the B field vanishes exactly. In general this solution is only an approximate solution to Maxwell's equations because Ampère's law (and of course the Ampère-Darwin law) rather than the Ampère-Maxwell law is satisfied outside the coils. An exact solution is obtained when the current is linear in time. The electric field is then constant in time.

For a current linear in time the result (a) by Griffiths and Heald applies, and we find a quasistatic solution for the electromagnetic fields that is also an exact solution. The condition of linear time dependence is in general not necessary; in fact there are quasistatic solutions with general time dependence that also solve Maxwell's equations as is illustrated by the next example.

Example 2. Any spherically symmetric solution of Maxwell's equations also solves the EQS and the Darwin models. Consider a spherically symmetric charge and current density satisfying the continuity equation (3). By spherical symmetry we obtain $\nabla\times\mathbf{E}=\nabla\times\mathbf{B}=0$. The solution to the EQS, Darwin, and Maxwell's equations is $\mathbf{E}=\mathbf{E}_C$, $\mathbf{B}=0$, and $\mathbf{J}=-\varepsilon_0\partial\mathbf{E}_C/\partial t$, which is valid for any charge density $\rho=\rho(r,t)$ in Eq. (5). The equivalences (44) and (47) apply with $\mathbf{a}=\mathbf{b}=0$. A particular case is a spherically symmetric leaking capacitor. We assume that the medium in between the spherical conducting shells has nonzero conductance. In between the shells there is zero charge density during the slow discharge. Ampère's law is clearly not satisfied because $\mathbf{B}=0$ while $\mathbf{J}\neq 0$.

The next three examples illustrate that in a linear, isotropic, and homogeneous conductor, the conduction current (assuming Ohm's law) does not create a magnetic field in certain cases. The corresponding solutions of Maxwell's equations also solve the EQS and Darwin models.

Example 3. Consider a charge distribution without symmetry. Space is assumed to be linear, isotropic, conducting, and homogeneous and characterized by $(\varepsilon_0,\mu_0,\sigma)$. Assume that at time $t=0$ we know the charge density $\rho(\mathbf{r},0)=\rho_0(\mathbf{r})$. From $\mathbf{J}=\sigma\mathbf{E}$ in the continuity equation and Gauss' law we find

$$\rho(\mathbf{r},t)=e^{-(\sigma/\varepsilon_0)t}\rho_0(\mathbf{r}). \tag{51}$$

The corresponding solution to Maxwell's equations is

$$\mathbf{E}(\mathbf{r},t)=e^{-(\sigma/\varepsilon_0)t}\mathbf{E}_0(\mathbf{r}) \text{ and } \mathbf{B}=0, \tag{52}$$

where $\mathbf{E}_0(\mathbf{r})$ is the Coulomb field due to the charge density $\rho_0(\mathbf{r})$.

Example 4. Example 3 may be modified so that we consider a finite conductor surrounded by free space. Let the conductor contain all the charge $\rho_0(\mathbf{r})$ at time $t=0$ and let the surface of the conductor be an equipotential surface with potential $V_0(\mathbf{r})$ defined by

$$V_0(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int\int\int\frac{\rho_0(\mathbf{r}')}{R}\,d\tau'. \tag{53}$$

Inside the conductor both the charge distribution and the solution to Maxwell's equations are the same as in Example 3. Outside the conductor the fields are static with $\mathbf{E}=\mathbf{E}_0(\mathbf{r})=-\nabla V_0$ and $\mathbf{B}=0$. The surface charge density on the conductor is

$$\sigma_S(\mathbf{r},t)=\varepsilon_0(1-e^{-(\sigma/\varepsilon_0)t})\hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{E}_0(\mathbf{r}), \tag{54}$$

where $\hat{\mathbf{n}}$ is the outward normal.

Example 5. Related to Example 4 is the leaking capacitor of arbitrary geometry and with homogeneous, isotropic, and weakly conducting material in between the two perfect conductors. In this case we must define $V_0(\mathbf{r})$ by Laplace's equation with appropriate boundary conditions on the two perfect conductors. Then the solution may be expressed in terms of $\mathbf{E}_0(\mathbf{r})=-\nabla V_0$ as in Examples 3 and 4. No magnetic field is created during the discharge of the capacitor.[33]

V. A QUASISTATIC PERSPECTIVE

The time-varying current of a long solenoid causes an induced electric field outside a solenoid. But how can that be? There is no magnetic field outside. A. P. French asked a similar question[3] that was addressed in Refs. 4–7. The answers included two essential points: (1) The vanishing of the B field is a static phenomena and in the nonstatic case there is a small magnetic field outside the solenoid. (2) It is not a good idea to consider the time-varying magnetic field as a source for the induced electric field; rather the source is the time-varying current.[14]

Even though both these statements are correct in view of Maxwell's equations, our understanding might still be incomplete. The way we calculate in practice the induced electric field from Faraday's law makes it natural to think in terms of the time-varying B field as a source and, even if there is a small B field outside the solenoid, the main part of this source is well separated in space from the effect we consider, that is, the induced outside E field. This apparent action at a distance seems to contradict the local action quality of Maxwell's equations. A careful analysis shows that there is no contradiction.

Let us now consider this problem from a different point of view. In the solenoid example it is easy to calculate the magnetic and electric fields using quasistatics. We obtain an exact solution within the quasistatic model and a good approximation to the true Maxwell solution. This quasistatic solution (of MQS and Darwin) has no magnetic field outside the solenoid. This solution is fundamentally inconsistent with Maxwell's equations because there is a time-varying E field in free space where the magnetic field vanishes exactly. However, the solution is consistent with quasistatics. The inconsistency with Maxwell's equations is an indication of the different nature of quasistatic models and Maxwell's equations. The conceptual problems have their roots in using quasistatic calculations, but interpreting the result using Maxwell's equations.

VI. ELECTROMAGNETIC SIMULATIONS

A. The computer in basic courses

Simulations are essential to electrical and electronic product designs. A broad range of important applications are within the quasistatic regime, including motors, sensors, power generators, transformer systems, and microelectromechanical systems. Industrial and scientific applications may involve extensive calculations and accordingly the need for large computational resources. The personal computer is a sufficiently powerful tool for modeling many electromagnetic phenomena, which makes it potentially useful for teaching electrodynamics. Examples within statics and quasistatics include edge effects in the parallel plate capacitor, distributed currents in conductors of various shapes, the Hall effect, magnetoresistance, the appearance of nonvanishing charge density in inhomogeneous conductors, the electric field outside and surface charge on a conductor with currents, various objects placed in a given external static or time-varying electromagnetic fields, eddy currents, inductive heating, magnetic diffusion, and magnetic shielding.[34]

The use of electromagnetic simulations as a pedagogical tool presupposes some support from theory. In particular, we should focus more on how Maxwell's equations are used to formulate well-posed partial differential equations for various physical situations. Most textbooks are not yet influenced by the use of computers. Their guiding principle is to find analytical solutions in simple and instructive cases. The underlying numerical problem may not be needed explicitly in some examples such as when integral formulations are used in symmetrical cases or when techniques like mirror charges or mirror currents work. When the partial differential equations are used, we find solutions in terms of integrals and series by comparatively tedious calculations involving, for example, special functions and variable separation.

If a numerical PDE solver is available, the situation is different. The formulation of well posed PDE (partial differential equation) problems for various phenomena is now motivated without the need to find analytical solutions. This formulation is an easy part of electromagnetics, and it provides greater insight into the structure of Maxwell's equations. It is in particular important and sometimes straightforward to find how Maxwell's equations reduce due to various symmetries. For example, the independence of one Cartesian coordinate or alternatively an axial symmetry results in a decoupling of Maxwell's equations. A simple inspection of the equations written in component form reveals this structure. Such symmetries explain, for example, why the magnetic field of a toroidal coil only has an azimuthal component or why there are TE and TM modes in planar waveguides. We illustrate in the following how symmetries reduce Maxwell's equations by formulating PDE problems for eddy currents.

B. Equations for eddy currents

Eddy currents and associated phenomena such as inductive heating, magnetic shielding, and magnetic diffusion are common because the basic ingredients are just a conductor and a time-varying electromagnetic field. Still eddy currents are not much discussed in textbooks. An exception is Ref. 35. The method of images as developed by Maxwell and reformulated in modern terms by Saslow[36] can be used to calculate eddy currents in a thin conducting sheet. However, the analytical theory for eddy currents is still difficult. Electromagnetic modeling is an attractive and easy way to include more about eddy currents in basic courses.

Eddy current problems fall into two classes, steady state and transient. In steady-state analysis (also called time-harmonic analysis) we replace $\partial/\partial t$ by $i\omega$ and allow for complex valued fields. Maxwell equations becomes time independent and much easier to solve. Physically we may obtain a steady-state condition at a time sufficiently long after the start of a time-harmonic source of the fields. The initial transient behavior is not considered in this analysis. The steady-state Maxwell equations may be further simplified in quasistatic situations. This simplification might be useful for analytical theory, but it is not of much interest for simulations. The situation is different for a time-dependent (transient) analysis. Then an initial value PDE must be solved and a quasistatic approximation may imply a major numerical simplification changing a hyperbolic PDE into a parabolic or elliptic one.[10–12]

Let us assume that the time variation is slow enough so that quasistatic theory applies. But what quasistatic model should we use in the study of eddy currents, EQS, MQS, or Darwin? Not EQS because magnetic induction (Faraday's law) is outside the scope of this model. MQS seems to be the standard model for the calculation of eddy currents, but it may only be used in some special cases.[13,35,37] A common misunderstanding concerning the quasistatic approximation may partly explain the popularity of MQS.[23]

We now consider two situations where the use of MQS may be justified because the symmetries cause MQS and Darwin to become equivalent models. The first one includes as a special case when Maxwell's theory for eddy currents in thin conducting sheets applies.[36]

Example 6. Consider a body in which the conductivity $\sigma = \sigma(z)$ varies only in the z direction. In the x and y directions the body is homogeneous and of infinite extent. The electromagnetic field is created by an external current density $\mathbf{J}^e = J_x^e(t,x,y,z)\hat{\mathbf{x}} + J_y^e(t,x,y,z)\hat{\mathbf{y}}$ with vanishing divergence, $\nabla \cdot \mathbf{J}^e = 0$. In this example there is no charge density and the Darwin model is equivalent to MQS. The reason is that electric fields will appear only in the directions of constant conductivity, that is, they have no z component, and they cause no pile up of charge. The system may be described in terms of the potentials A_x and A_y with $V = A_z = 0$. From Eq. (24) with $\mathbf{J} = \mathbf{J}^e + \sigma(-\nabla V - \partial \mathbf{A}/\partial t)$ we obtain

$$-\mu_0 \sigma \frac{\partial A_x}{\partial t} + \nabla^2 A_x = -\mu_0 J_x^e, \tag{55}$$

$$-\mu_0 \sigma \frac{\partial A_y}{\partial t} + \nabla^2 A_y = -\mu_0 J_y^e. \tag{56}$$

These three-dimensional equations determine the dynamics and may be used to model physical systems with the above symmetry.

Example 7. A similar two-dimensional axisymmetric case may also be formulated. We use cylindrical coordinates (r,ϕ,z) and an axially symmetric conducting body with conductivity $\sigma = \sigma(r,z)$. The external current density is $\mathbf{J}^e = J^e(t,r,z)\hat{\phi}$. In this case there will be no charge density, $V = A_r = A_z = 0$, and the system may be described in terms of the vector potential $\mathbf{A} = A(t,r,z)\hat{\phi}$. The equation for the dynamics is

$$-\mu_0\sigma\frac{\partial A}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{\partial^2 A}{\partial z^2}=-\mu_0 J^e. \tag{57}$$

Equation (57) may be studied using a PDE solver.[34] For example, a circular copper wire above a circular copper plate is considered by Backstrom[38] using Eq. (57) and the solver FlexPDE. The transient fields appearing when a current is turned on in the wire is studied (Ref. 38, p. 99). The models in Examples 6 and 7 can also be studied using Comsols Mulphysics with the Electromagnetic Module. These problems are numerically simple and there are many other PDE solvers that may be used.

These examples were designed in order to avoid the charge density as is the case for all the eddy current examples in Ref. 36 (see Appendix B). Usually there will appear time-varying surface charge on conductors with eddy currents, and, in an inhomogeneous conductor, there will also appear charge density inside the conductor. To describe the physics we need to include the scalar potential in the analysis and MQS cannot be used. An interesting possibility is to use the Darwin model. Such applications of Darwin's model have been suggested in connection with eddy currents in the human body caused by high voltage transmission lines.[39] The traditional use of Darwin's model is not eddy currents, but concerns charged particle beams and plasma simulations. Let us write equations for the Darwin model in term of potentials. The conductivity of a possibly inhomogeneous conductor is $\sigma=\sigma(\mathbf{r})$ and the time-dependent external current $\mathbf{J}^e=\mathbf{J}^e(t,\mathbf{r})$ is prescribed. The total current may be written as $\mathbf{J}=\mathbf{J}^e+\sigma(-\boldsymbol{\nabla}V-\partial\mathbf{A}/\partial t)$. We use, as always in the Darwin model, the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A}=0$. The dynamics of the potentials is determined by the continuity equation

$$-\varepsilon_0\frac{\partial}{\partial t}\nabla^2 V-\boldsymbol{\nabla}\cdot(\sigma\boldsymbol{\nabla}V+\sigma\partial\mathbf{A}/\partial t)=-\boldsymbol{\nabla}\cdot\mathbf{J}^e, \tag{58}$$

and the Ampère-Darwin equation is

$$-\varepsilon_0\mu_0\frac{\partial}{\partial t}\boldsymbol{\nabla}V-\mu_0(\sigma\boldsymbol{\nabla}V+\sigma\partial\mathbf{A}/\partial t)+\nabla^2\mathbf{A}=-\mu_0\mathbf{J}^e. \tag{59}$$

These equations have a nonstandard appearance because they contain mixed time and spatial derivatives. They may be solved using Comsol Multiphysics. There is a considerable freedom for the user to enter nonstandard equations in this PDE solver. This freedom follows from the option to formulate the equations in the weak form, the natural form of equations in the finite element method. The numerical solutions appears, at least qualitatively, to behave in an expected way for the few examples that we have considered. The practical usefulness of the Darwin model for eddy current calculations remains to be seen.

A qualitatively new feature of Darwin's model, in contrast to MQS or EQS, is the possibility of resonance. Because the Darwin model includes both capacitive and inductive phenomena, it can in principle be used to model systems where the energy oscillates between the electric and magnetic fields. Consider, for example, a field theoretic version of a LC circuit. A suitable design for such an application may be a resonator in the form of a short coaxial cable where one end is short circuited by a metal plate and the other end is still electrically open, but with increased capacitance created by the use of two close parallel plates connected to the inner and outer conductor, respectively. This kind of resonator has applications in connection with electron beam devices at microwave frequencies (see Example 3.4.1 in Ref. 9). The resonance frequency of a LC circuit is $f=1/\sqrt{LC}$, but the frequency must be low enough not to violate the quasistatic assumption. In the above design of a resonator this assumption can be satisfied by making the capacitance large, thus taking the two capacitor plates very close to each other.

VII. SUMMARY

Maxwell's equations are fundamental for the description of electromagnetic phenomena and valid over a wide range of spatial and temporal scales. The static limit of the theory is well defined and much easier. The electric and magnetic fields are given by the laws of Coulomb and Biot-Savart. As soon as there is any time dependence, we should in principle use the full set of Maxwell's equations with all their complexity. Time retardation is a fundamentally important and also a complicating feature. Even if the effect is small, it will not vanish, which makes the theory unnecessarily complicated. In numerical analysis these effects, however small, may force us to use smaller time steps for numerical stability. It is therefore useful to introduce quasistatic approximations in Maxwell's equations. The quasistatic models are also useful for a better understanding of both low frequency electrodynamics and the transition from statics to electrodynamics.

The major points of this paper include the following: (1) The quasistatic limit of Maxwell's equations is a kind of $c\to\infty$ limit obtained by neglecting time retardation. The Darwin model is obtained if we use the Coulomb gauge. (2) The Darwin model involves both capacitive and inductive features, but there is no radiation and the interactions are instantaneous. Poynting's theorem for this model shows that there is both electric and magnetic energy, but the electric energy only includes the Coulomb part of the electric field. (3) The EQS and MQS models may be considered as approximations of the Darwin model. Poynting's theorem for these models shows that there is only electric energy in EQS and only magnetic energy in MQS. (4) The law of Biot-Savart is valid within EQS, MQS, and the Darwin model. (5) Coulomb's law is of general significance for quasistatics as is obvious when we use the formulation in terms of integral expressions (Sec. II A) or use potentials (Sec. II D). (6) Ampère's law is not of general significance in quasistatics. It is valid only in MQS, but not in EQS and Darwin.[23] (7) The Galilean invariance of quasistatics is a delicate issue. Galilean invariance may be defined for EQS and MQS, but if we want the force to have the corresponding invariance, then only the electric force is included in EQS and only the magnetic force in MQS.[30] (8) Quasistatics has important applications in the electromagnetic modeling of transient phenomena.

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[14] It has been tempting to regard $\partial \mathbf{B}/\partial t$ in Faraday’s law as a source of an electric field, thereby sometimes causing confusion and objections; currents and charges should be considered as the only sources of electromagnetic fields. The integrals expressing the fields in terms of these sources involve, in general, the retarded time so that the news from the sources propagates with the finite velocity c. In a quasistatic approximation the interactions are instantaneous so this objection is less valid.
[15] The static continuity equation together with Eqs. (1) and (2) constitute a quasistatic model without capacitive or inductive effects.
[16] The integral (6) converges as is evident from the law of Biot-Savart, which implies that the B field goes to zero as $1/r^2$ when $r \to \infty$.
[17] We consider systems where the charges and currents do not extend to spatial infinity and may use as boundary conditions that the fields approach zero sufficiently fast far away.
[18] Ampère’s law, $\nabla \times \mathbf{H} = \mathbf{J}_f$, differs from Eq. (15) if there is a nonvanishing polarization current $\partial \mathbf{P}/\partial t$. Ampère’s law is, strictly, speaking, a static equation, and we prefer to use Eq. (15) as a time-dependent generalization with the same name.
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