Experimental determination of entanglement by a projective measurement

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We describe a method in which the entanglement of any pure quantum state can be experimentally determined by a simple projective measurement, provided the state is available in a twofold copy. We propose implementations of this approach for various systems and discuss in detail its first experimental realization, which employed twin photons entangled in two degrees of freedom, prepared in identical polarization and momentum states. We analyze the effect of errors due to imperfect state preparation.

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I. INTRODUCTION

Entanglement, one of the fundamental characteristics which distinguishes the quantum from the classical world and a key resource for quantum information processing, remains a puzzling and still counterintuitive concept when it comes to its quantification. In principle, all we can know about a given quantum state is encoded in its statistical operator, and so is entanglement, but nonlinearly so [1,2]. As a consequence, we cannot immediately infer the state's entanglement from simple inspection of the density matrix elements, but rather have to evaluate nonlinear functions thereof. Thus, quantum-state tomography [3–5], which allows, at least in principle, for a complete measurement of all elements of the density matrix, appears as the only reliable strategy towards the complete description of the nonclassical correlations inscribed into an arbitrary state.

However, quantum-state tomography rapidly reaches its practical limits as the dimension of the density matrix increases with the number and/or the Hilbert space dimensions of the constituent subsystems. Indeed, the experimental determination of the entanglement of a system composed of more than a few subsystems using tomographic means seems elusive given the unfavorable scaling of the required experimental resources.

Furthermore, existing entanglement measures lack direct physical meaning, since they are obtained by applying state transformations like conjugation and partial transposition, which are not completely positive [1,2].

Therefore, alternative, physically motivated and less demanding strategies to determine or estimate the entanglement of composite quantum systems of increasing size would be quite useful, the more so since experiments nowadays achieve ever improving state control on ever increasing multipartite quantum systems.

Entanglement witnesses [6] provide one such alternative and have proven both versatile and efficient in the detection of specific classes of entangled states. However, witnesses are state specific; i.e., they are constructed to detect specific types of entangled states and fail to identify others. Therefore, their successful use relies on some *a priori* knowledge of the state under scrutiny.

We have recently devised another alternative for the efficient determination of the degree of entanglement of a pure state, which is based on a projective measurement. If the state is available in a twofold copy [7,8], then the state's concurrence [1,9] is given as the expectation value of a single, suitably defined, self-adjoint operator A, defined with respect to the twofold copy. Apart from the availability of two faithful copies of a pure state, no further assumptions are necessary. In particular, A is *state independent*, and the very same measurement protocol is applicable for *arbitrary* pure states, in arbitrary dimensions.

In the present paper, we specialize the general theoretical treatment to the exemplary case of entangled twin photons, which allow a simple demonstration of the efficiency of this approach in the laboratory. The experimental realization is discussed in detail, with an evaluation of possible sources of error due to experimental inaccuracies in the initial state preparation. We also show how our method could be implemented in cavity quantum electrodynamics (cavity QED) and with trapped ions.

II. THEORETICAL IDEA

The original definition of concurrence—restricted to systems of two qubits described by a pure state $|\Psi\rangle$ in the Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$—reads [1]

$$c(\Psi)=|\langle\Psi^*|\sigma_y\otimes\sigma_y|\Psi\rangle|, \tag{1}$$

where σ_y is the second Pauli matrix and $\langle\Psi^*|$ is the complex conjugate of $\langle\Psi|$, with the conjugation performed in the stan-

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dard product basis. This is *different* from the expectation value of the Hermitian operator $\sigma_y \otimes \sigma_y$ with respect to the state $|\Psi\rangle$, due to the complex conjugation of the bra. Equation (1) does not define a valid measurement prescription.

Nevertheless, it *is* possible to express concurrence in terms of a legitimate projective measurement: Indeed, one can rewrite concurrence in terms of an overlap of a twofold copy of $|\Psi\rangle$—i.e., $|\Psi\rangle \otimes |\Psi\rangle$—with some state that is an element of the duplicate Hilbert space $\mathcal{H} \otimes \mathcal{H}$—i.e., of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$. The algebraic identity

$$\langle \Psi^* | \sigma_y \otimes \sigma_y | \Psi \rangle = 2 \langle \chi | (|\Psi\rangle \otimes |\Psi\rangle) \tag{2}$$

holds for $|\chi\rangle = (|0011\rangle - |0110\rangle - |1001\rangle + |1100\rangle)/2$—that is, for the *unique* state that is antisymmetric with respect to the exchange of the two copies of the first subsystem (the first and the third entries of $|\chi\rangle$), as well as with respect to the exchange of the two copies of the second subsystem (the second and fourth entries). The state $|\chi\rangle$ is simply the product of the antisymmetric state $|\chi_1\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ of the duplicate first subsystem and its analogous counterpart $|\chi_2\rangle$ for the second subsystem. For a separable state $|\Psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$, one easily verifies

$$\langle \chi_i | (|\phi_i\rangle \otimes |\phi_i\rangle) = 0 \quad (i = 1,2), \tag{3}$$

since $|\phi_i\rangle \otimes |\phi_i\rangle$ is symmetric for any state $|\phi_i\rangle$. Thus, it becomes evident that concurrence vanishes for any separable state. Furthermore, it is also strictly positive for any nonseparable state [10].

From Eqs. (1) and (2) it follows that

$$c(\Psi) = \sqrt{\langle \Psi | \otimes \langle \Psi | A | \Psi \rangle \otimes |\Psi\rangle}, \tag{4}$$

with $A = 4|\chi\rangle\langle\chi|$. Consequently, concurrence may be expressed as the expectation value of a Hermitian operator and may thus be related to a measurable quantity.

Moreover, the formulation of Eq. (4) also has a nice generalization to arbitrary finite dimensions, where $|\chi\rangle\langle\chi|$ is replaced by $P_-^{(1)} \otimes P_-^{(2)}$, the direct product of the projectors onto the antisymmetric parts of the duplicate first and second subsystems. These projectors are defined in arbitrary dimensions—the only difference from the case of qubits is that they are not one dimensional anymore, which reflects the fact that there is no single state $|\chi\rangle$ that distinguishes all entangled states from the separable ones. With the definition of concurrence given in Eq. (4) and $A = 4P_-^{(1)} \otimes P_-^{(2)}$, one not only ensures that c is nonvanishing for all entangled states, but this expression is also equivalent to a prior definition of concurrence [11],

$$c(\Psi) = \sqrt{2(1 - \mathrm{Tr}\varrho_1^2)} = \sqrt{2(1 - \mathrm{Tr}\varrho_2^2)}, \tag{5}$$

in terms of the purity of the reduced density matrices $\varrho_1 = \mathrm{Tr}_2|\Psi\rangle\langle\Psi|$ or $\varrho_2 = \mathrm{Tr}_1|\Psi\rangle\langle\Psi|$—tantamount to the information loss on the state of one subsystem (1 or 2) upon tracing over its complement (2 or 1).

Equation (4) defines a measurement prescription in terms of the Hermitian operators $P_-^{(1,2)}$. This measurement does not involve collective observables on both the first and second subsystems, but only on the copies of either one. That is, concurrence is now expressed in terms of only separable observables—though on a twofold copy of the state $|\Psi\rangle$. At first sight, this procedure might seem to be incompatible with the no-cloning theorem [12,13]. However, this is *not* the case: since in any experiment a state has to be prepared repeatedly in order to obtain reliable measurement statistics, the only step one has to take beyond typical experimental techniques is to replace the individual measurements on each of the separately prepared copies of a state by a combined measurement on pairs of those, which should be available at the same time. It has been known for some time that combined measurements, performed on two copies of a state, can be more efficient, in terms of information transfer, than individual measurements [14]. Also, it is in general possible to measure polynomial functions of the elements of a density matrix by finding the expectation value of an observable on several copies of the density matrix, without performing state tomography [15–17]. We discuss in the following how such a collective observable leads to a direct measurement of concurrence, which is directly implementable in various physical systems with relative ease.

Equation (5) shows that the degree of mixing of ϱ_1 equals that of ϱ_2. Therefore, concurrence can be expressed in terms of either of those. An analogous choice is left open in the explicit formulation of Eq. (4). Indeed, let P_+ be the projector onto the symmetric space, with $P_+ + P_- = \mathbb{1}$. The products $P_-^{(1)} \otimes P_+^{(2)}$ and $P_+^{(1)} \otimes P_-^{(2)}$ project onto states that are globally antisymmetric—i.e., antisymmetric with respect to an exchange of the two copies of the *entire* bipartite system. The twofold copy of the system state, however, is globally symmetric. Therefore, the expectation values of both of these operators necessarily vanish with respect to a twofold copy of any state $|\Psi\rangle$. Consequently, one can also measure $P_-^{(1)} \otimes P_-^{(2)} + P_-^{(1)} \otimes P_+^{(2)} = P_-^{(1)} \otimes \mathbb{1}^{(2)}$ or $P_-^{(1)} \otimes P_-^{(2)} + P_+^{(1)} \otimes P_-^{(2)} = \mathbb{1}^{(1)} \otimes P_-^{(2)}$, instead of $P_-^{(1)} \otimes P_-^{(2)}$. The expectation values of these three operators coincide on a twofold copy of $|\Psi\rangle$. In a single run of an experiment, this simply reflects that, if a measurement on one of the duplicate subsystem reveals an antisymmetric or symmetric state, then the other duplicate subsystem is projected onto an antisymmetric or symmetric state. Consequently, concurrence can be measured with a single measurement on only one of the twin subsystems, as long as one deals with pure states. Experiments with two-photon entangled states obtained from parametric down-conversion, for example, have demonstrated a high degree of purity [18]. In other systems, where the degree of purity of the state might be smaller, a measurement on the second duplicate subsystems may be employed to experimentally verify the purity of the state. If the system is in a mixed state ϱ, its purity is easily related to P_- and P_+ via $1 - \mathrm{Tr}\varrho^2 = 2\mathrm{Tr}[\varrho^{\otimes 2}(P_-^{(1)} \otimes P_+^{(2)} + P_+^{(1)} \otimes P_-^{(2)})]$ [19].

III. EXPERIMENTAL IMPLEMENTATIONS

The projector P_- is, in principle, measurable in any composite quantum system. While in many systems it is technically very demanding and often virtually impossible, to measure a complete set of observables, it is indeed rather easy to measure P_- in various systems. In particular, for two-level systems, P_- is simply the projector onto the singlet state $|\psi^-\rangle=(|01\rangle-|10\rangle)/\sqrt{2}$.

For photon pairs, the most prominent experimental method to obtain the probability of projecting two photons in the singlet state is by using a Hong-Ou-Mandel interferometer: the two photons are injected from different sides of a beam splitter, and one measures the fraction of the total number of photons that leave the beam splitter in different momentum modes [20], since for the three other Bell states both photons come out along the same direction. This measurement would be applied on two photons belonging to two different (entangled) photon pairs, each pair generated by a nonlinear crystal.

An analogous effect can also be observed with electron-spin systems in solid-state devices, where the probability to observe $|\psi^-\rangle$ can be measured simply in terms of current correlations [21], or within the framework of cavity QED [22] and trapped ions [23,24]. Cavity QED can be employed for the measurement of concurrence of a two-atom entangled state, repeatedly produced in some experimental setup. In the first step, one transfers the atomic state to a two-mode cavity field, initially in the vacuum state $|00\rangle$. This can be done by letting each of the atoms in the entangled state be resonant with one of the modes, so that if the atom enters the cavity in the upper state $|e\rangle$, it leaves one photon in the corresponding mode, exiting the cavity in the lower state $|g\rangle$:

$$(\alpha|ee\rangle+\beta|eg\rangle+\gamma|ge\rangle+\delta|gg\rangle)\otimes|00\rangle \rightarrow |gg\rangle\otimes(\alpha|11\rangle+\beta|10\rangle+\gamma|01\rangle+\delta|00\rangle). \quad (6)$$

Next, one sends a second set of entangled atoms, prepared in the same state as the first one, through the cavity. The aim now is to determine the probability of finding one of the atoms, together with one of the cavity modes, in the Bell state $|\psi^-\rangle=(|g1\rangle-|e0\rangle)/\sqrt{2}$. This can be accomplished by using the technique described in [25], which proposed a scheme for teleportation of atomic states using two cavities. The result yields the concurrence of the two-atom state.

For trapped ions, a possible experimental procedure would consist in first producing two copies of an entangled state on two pairs of ions in a trap, using the internal degrees of freedom of the two ions. One would then map the entangled state of one of the pairs onto two collective vibrational modes (involving the four ions), and finally one would measure the Bell states corresponding to the internal degrees of freedom of one of the atoms of the second pair and one of the vibrational modes. This measurement can be made by implementing a controlled-NOT gate between the ion internal state and one of the vibrational modes (see Refs. [23,24] for the implementation of controlled-NOT gates in ion traps) and then applying a $\pi/2$ rotation on the internal atomic state. Measurement of the atomic-state and vibrational-state populations, which can be done by standard techniques, yields direct information on the Bell states. Though it is possible in principle to implement the required projective measurement, application of this scheme to cavity QED and trapped ions would require the preparation of highly pure states, which is not an easy task with present technologies. Twin-photon systems are privileged in this respect, since purities of the order of 99% have been reached in recent experiments [18].

IV. EXPERIMENTAL REALIZATION WITH TWIN PHOTONS

Among other systems, twin photons generated by spontaneous parametric down-conversion allow for an efficient experimental demonstration of the two-copy scheme. Projection of two photons onto the antisymmetric Bell state using Hong-Ou-Mandel interference has been performed in the context of teleportation [26] and entanglement-swapping experiments [27,28]: Two pairs of photons were produced via parametric down-conversion using the "double-pass" geometry, in which a laser pulse propagates through the nonlinear crystal source and is then reflected by a mirror back through the crystal. When one pair of photons is produced in each interaction, two photons (one photon of each pair) are incident simultaneously on different sides of a 50-50 nonpolarizing beam splitter. The registration of a two-photon coincidence count of one photon in each output mode of the beam splitter signals the measurement of the antisymmetric Bell state. In [27,28], two photons taken from two different copies of a maximally entangled state were projected onto the antisymmetric state, as is required to determine the concurrence using Eq. (4). However, only maximally entangled states were used in [27,28], as this maximizes the amount of entanglement to be swapped. In contrast, Eq. (4) provides a recipe to directly infer the concurrence of *arbitrary* pure states. Since this calls for an expectation value, also the registration of all possible final states of one photon and its copy—symmetric as well as antisymmetric Bell states—is required for proper normalization.

Finally, to create two perfect copies of polarization-entangled photons one would have to prepare the first pair, store them, and prepare the second pair with the *same* preparation system. Thus, to perform the concurrence measurement using two-photon interference, one would need heralded photon pairs (i.e., the availability of the photons is marked by a trigger signal), as well as a precision photon memory device, to assure that the photons of different pairs "meet" at the beam splitter. However, present-day parametric down-conversion sources are probabilistic. To overcome the need for heralded photons and storage one could create two pairs simultaneously, one photon in each of four different spatial modes, as in Refs. [26–28], and use different preparation devices for each entangled photon pair. In this case the copies are not perfect, since there will be errors due to imperfect calibration and alignment of the different preparation systems.

An essentially equivalent method is to generate two photons instead of four and utilize two different degrees of freedom for each photon [29]—polarization and momentum, for example—to store two copies of the state. This significantly simplifies the measurement apparatus, as a complete Bell-state measurement of polarization-momentum states can then be realized with perfect efficiency using linear optics only [30]. Recently, we reported the experimental determination of concurrence using this encoding scheme [8]. The experimental setup is depicted in Fig. 1.

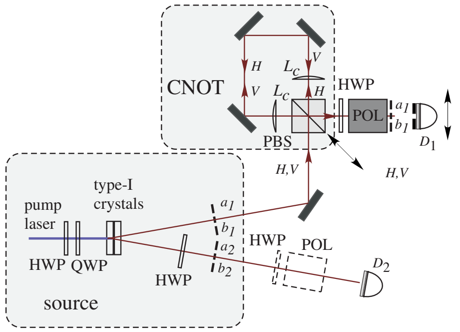

FIG. 1. (Color online) Experimental setup for entanglement measurement on two copies of the quantum state. Photons entangled in both polarization and momentum are produced by pumping two nonlinear crystals with a laser beam and detected by detectors D_1 and D_2. A CNOT gate, where the change of the momentum of one photon is conditioned on its polarization, is implemented with a polarization-sensitive beam splitter (PBS) and two cylindrical lenses L_c.

A. Preparation

Hyperentangled states—i.e., states which bear simultaneous entanglement in two or more different degrees of freedom—are routinely prepared with parametric down-conversion [29,31]. Instead of pumping a single crystal, one can use two crystals, aligned with their optical axes perpendicular. If one of the crystals generates horizontally polarized photons, then the other yields pairs of vertically polarized photons. The pump laser generates a pair of photons that may be horizontally or vertically polarized, depending on the crystal that generated the pair. These two possibilities are coherent as long as the coherence length of the pump laser is larger than the length of the crystals. In this case, the state of the photon pair is $|\Psi_0\rangle = \alpha|H\rangle|H\rangle + \beta|V\rangle|V\rangle$, where H and V stand for horizontal and vertical linear polarization, respectively. The complex coefficients α and β are directly determined by the polarization state of the pump beam, which can be controlled using a half- and quarter-wave plate [3,18]. Using additional waveplates in the paths of the down-converted photons, one can create even more general states. For example, a half-wave plate placed in the path of photon 2 yields the state $|\Psi\rangle = \alpha|H\rangle|V\rangle + \beta|V\rangle|H\rangle$.

In addition to the polarization, the momentum degrees of freedom can be used to encode a copy of the pure state $|\Psi\rangle$. Contrary to polarization entanglement, down-converted photons are naturally entangled in momentum as a consequence of the phase matching conditions, and thus it is also possible to use this same source to create states which are entangled in momentum [29,32]. In fact, it is difficult to produce twin photons that are not entangled in momentum. Two distinct spatial modes a and b can be selected for each photon using a double aperture. Due to the emission cones of the down-converted photons, it is more convenient to use a vertically aligned double aperture, as all the collected photons have approximately the same frequency spectrum [18]. The experiment reported in Ref. [8] used two vertically aligned 1-mm^2 squares separated by 1 mm (the apertures shown just after the crystals in Fig. 1 were drawn in the horizontal for convenience). The magnitude of the coefficients α and β of a state $|\Psi\rangle = \alpha|a\rangle|b\rangle + \beta|b\rangle|a\rangle$ can be controlled by changing the position of the double apertures, since the momentum spatial distribution has a Gaussian profile. Alternatively, neutral density filters placed in front of the apertures could be used to control the relative weights of α and β. The relative phase between α and β can be controlled by tilting a thin glass plate in one of the modes.

The combination of these two degrees of freedom allows for easy preparation of entangled states such as

$$|\Psi\rangle \otimes |\Psi\rangle = (\alpha|HV\rangle + \beta|VH\rangle) \otimes (\alpha|ab\rangle + \beta|ba\rangle). \quad (7)$$

Thus, after identification of the momentum state $|a\rangle$ as the equivalent of the polarization state $|H\rangle$ and analogously for $|b\rangle$ and $|V\rangle$, one ends up with two copies of the same state $|\Psi\rangle$, stored in the respective degrees of freedom, of a *single* pair of photons. Equation (7) describes a general state in a given Schmidt basis. All other states can then be obtained by the application of suitable local unitary transformations (waveplates, etc.) or can be prepared directly using a slightly different experimental setup [33].

For the pure state in Eq. (7) one needs to assume an idealized preparation process. In any laboratory experiment, there is some residual mixing. The purity of the polarization state is limited by so called "which way information"—that is, the possibility of distinguishing from which crystal the photon pair was emitted renders the $|HH\rangle$ and $|VV\rangle$ components incoherent, which ruins the entangled state. Consequently, it depends on the spatial overlap between vertical and horizontal modes. It is possible to obtain arbitrarily pure states by spatial mode filtering, which is done by allowing the down-converted photons to propagate in free space (about a meter across the optical table is sufficient) and using pinholes to select low-order spatial modes. The same filtering can be achieved using lenses instead of free propagation [34]. In typical experiments, purities higher than 99% have been obtained [3,18]. In [8] the purity of the state, estimated by performing polarization interference measurements, was approximately 95%. Narrower crystals or narrower pinholes help to increase this value, at the expense of reducing the flux of detected photon pairs.

The two-crystal source has some advantages, in comparison to sources using a single type-II crystal [35]. With the two-crystal source, no longitudinal walk-off compensation is required, since any pair of photons have the same polarization and thus experience the same amount of temporal delay due to birefringence of the crystals. Coherence of the state is assured when the coherence length of the pump laser is larger than the width of the two-crystal ensemble. The phase of the state depends only on the phase difference between the pump field in each crystal and thus is very stable.

The purity of the momentum state depends essentially on the mode filtering, crystal length, and the spectral properties of the pump laser beam. Typical momentum interference experiments [32,36] report results corresponding to purities of approximately 90%, due to errors in temporal and spatial mode matching. Given the small apertures used to select the momentum modes, the narrow bandwidth interference filters, and the proximity of the momentum modes, as well as the

Sagnac interferometer configuration used in the experiment reported in Ref. [8], we estimate that the purity of the momentum state can be at least 90%.

B. Measurement

For the measurement of concurrence, one needs to find the probability that the state of a duplicate subsystem is observed in the singlet state $|\psi^-\rangle=(|01\rangle-|10\rangle)/\sqrt{2}$. For the two copies of the first subsystem in the specific setting of Ref. [8], this antisymmetric state reads $|\psi^-\rangle=(|Hb\rangle-|Va\rangle)/\sqrt{2}$. Thus, one single collective measurement in the Bell basis

$$|\phi^{\pm}\rangle=\frac{1}{\sqrt{2}}(|H\rangle\otimes|a\rangle\pm|V\rangle\otimes|b\rangle), \tag{8a}$$

$$|\psi^{\pm}\rangle=\frac{1}{\sqrt{2}}(|H\rangle\otimes|b\rangle\pm|V\rangle\otimes|a\rangle), \tag{8b}$$

comprised of both momentum and polarization, is required.

Measurements in entangled bases are typically far more involved than measurements in separable bases. Therefore, the task is significantly eased if the Bell states can be mapped onto separable states by a suitable transformation. As is well known, this can be achieved with a controlled-NOT (CNOT) gate [37,38], where the evolution of one subsystem is conditioned by the state of the second. In the present case, the evolution of the momentum state is controlled by the polarization, which can be realized by a polarization-sensitive Sagnac interferometer as depicted in Fig. 1. The input photons are first incident on a polarizing beam splitter (PBS), which transmits H-polarized photons and reflects V-polarized photons, so that H- and V-polarized photons propagate in opposite senses within the interferometer and leave through the same exit port. Cylindrical lenses are used to transform the momentum modes. Since the different polarizations propagate in opposite directions, the action of the cylindrical lenses L_c indeed depends on the polarization states, as is described in detail in the following section.

C. Cylindrical lenses

Figure 2 shows a set of situations in which an object is imaged with a cylindrical lens or lenses. Figure 2(a) shows the simplest case where the image is inverted in the horizontal direction (flipped around the vertical axis) and unchanged with respect to the vertical direction. This is the important difference in comparison with a spherical lens, where the image is inverted in both vertical and horizontal directions.

Figure 2(b) shows what happens when the object is rotated with respect to the cylindrical lens axis, which is vertical in Fig. 2(b). The image is again flipped around the vertical axis, which is equivalent to a rotation by twice the object angle and inversion with respect to the horizontal direction. In Fig. 2(b) we start with an image aligned at −45°, ending up with an inverted image aligned at +45° with respect to the vertical. Therefore, the result is a total rotation of 90°. If the initial object is a line "−," instead of an "F," the inversion has no physical effect and the image is equivalent to the object field rotated by twice the object angle.

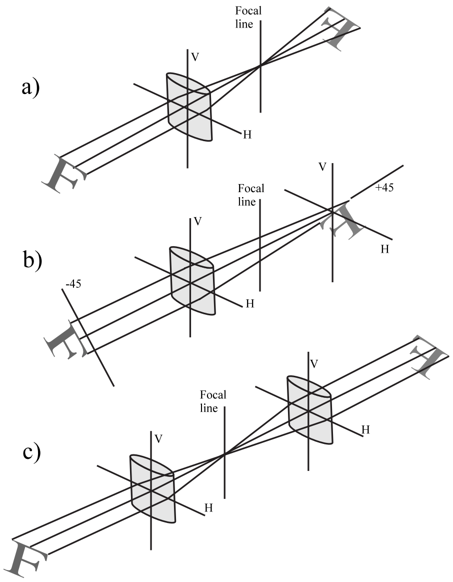

FIG. 2. Image formation by a cylindrical lens: (a) the image is flipped with respect to the vertical axis, (b) the image is rotated 90° with respect to the object, and (c) two confocal lenses are equivalent to a single lens.

In Fig. 2(c) we show that two cylindrical lenses in a confocal arrangement are equivalent to a single lens. This equivalence preserves the rotation rule: an object imaged by a double-lens system is rotated twice the angle between object and lens axis.

There are two more subtle points to the use of the cylindrical lenses within the interferometer. Since the photons travelling in the interferometer undergo an odd number of mirror reflections while propagating between the two cylindrical lenses, one also has to take into account the action of these mirrors when aligning the lenses. Figure 3 shows the effect of a mirror reflection on the propagation of two parallel beams. Considering a given propagation axis, the reflection inverts the positions of the beams with respect to the propagation direction. The one to the right of the propagation axis at the input will be on the left after the mirror reflection. Therefore, the effect of a mirror is to invert an image with respect to the normal plane. In the case of three mirrors

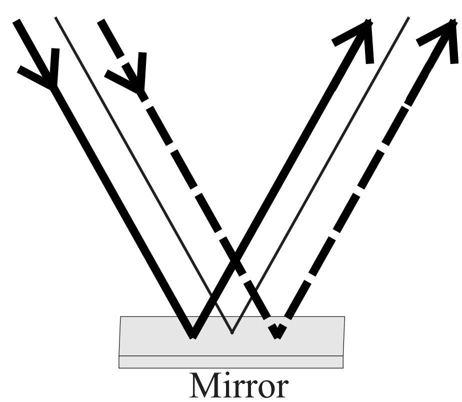

FIG. 3. Effect of reflection through a plane mirror.

between the two cylindrical lenses as in Fig. 1, the light field will be inverted with respect to the horizontal after three reflections. Thus, a light beam propagating in this interferometer will see the second cylindrical lens inverted with respect to the horizontal. For this reason it is necessary to align the cylindrical lenses at +45° and −45° with respect to the vertical axis. This inversion compensates the reflections at the three mirrors and the effective propagation of the light beam can be described by an arrangement like the one in Fig. 2(d).

The second subtlety is that, since (i) the H-polarized component of the field suffers one mirror reflection before reaching the lens system, while (ii) the V-polarized component of the field suffers two reflections before reaching the lens system, and (iii) the H- and V-polarized photons propagate in opposite directions within the Sagnac interferometer, the H and V polarization components encounter the cylindrical lens systems aligned at +45° and −45°, respectively. The momentum modes of the H-polarized photons are thus rotated by 90°, while the momentum modes of the V-polarized photons are rotated by −90°. The result is a relative 180° rotation, which is equivalent to a CNOT gate, defined by the operation

$$|H\rangle \otimes |a\rangle \to |H\rangle \otimes |\tilde{a}\rangle$$

$$|H\rangle \otimes |b\rangle \to |H\rangle \otimes |\tilde{b}\rangle$$

$$|V\rangle \otimes |a\rangle \to |V\rangle \otimes |\tilde{b}\rangle$$

$$|V\rangle \otimes |b\rangle \to |V\rangle \otimes |\tilde{a}\rangle,$$

where $\tilde{a}$ and $\tilde{b}$ denote the two momentum modes that are obtained from a and b by the overall 90° rotation caused by the interferometer. It is possible—but unnecessary—to rotate $\tilde{a}$ and $\tilde{b}$ back to the vertical axis using a second cylindrical lens system.

Eventually, this operation takes the Bell-states of Eq. (8) into separable states $|\phi^+\rangle \to |+\rangle \otimes |\tilde{a}\rangle$, $|\phi^-\rangle \to |-\rangle \otimes |\tilde{a}\rangle$, $|\psi^+\rangle \to |+\rangle \otimes |\tilde{b}\rangle$, and $|\psi^-\rangle \to |-\rangle \otimes |\tilde{b}\rangle$, where $|\pm\rangle = |H\rangle \pm |V\rangle$. The identification of all four polarization-momentum Bell states can then be performed by measuring the polarization and momentum in adequate bases.

D. Fine-tuning of the copies

For the present measurement approach it is vital to have two precise copies [39], such as Eq. (7), instead of

$$|\Psi\rangle \otimes |\Psi\rangle = (\alpha|HV\rangle + \beta|VH\rangle) \otimes (\alpha'|ab\rangle + \beta'|ba\rangle). \quad (9)$$

The prefactors α and β of the polarization state can be determined via an initial calibration of the setup without the interferometer. The absolute values of α' and β' can be obtained by the normalized coincidence counts of the detectors D_1 at positions a_1 and b_1.

The polarization and momentum states prepared in this fashion are nearly perfect copies of each other: they have the same Schmidt coefficients, and given the identification of local polarization and momentum bases, their Schmidt bases coincide—up to a difference of local phases. Since concurrence is a function of the Schmidt coefficients only, both copies do have the same value of concurrence, and, as we elaborate later in Sec. V A, the present measurement of concurrence is indeed insensitive to this difference of local phases.

It is important, however, that these phases have the same value for each experimental run, so that a pure state is effectively produced. The phase of the polarization state depends basically on the pump polarization state, which can be easily controlled. Other problems, like fluctuations of the refractive index of the crystals due to temperature changes, are insignificant. All experiments (see Refs. [3,18], for example) performed with entangled polarization states produced by this kind of source have shown phase stability. The phase of the momentum state depends basically on the propagation across the optical table. In our setup, different momentum modes propagate very close to each other and pass through the same optical components, due to the convenient Sagnac interferometer configuration. Therefore, any relative phase shifts are negligible. The phase stability of the interferometer in Ref. [8] was tested using first-order interference.

V. ERRORS DUE TO IMPERFECT PREPARATION

So far, we have assumed that the preparation of the twin states is perfect. Since a laboratory experiment can only be conducted with limited precision, we discuss here the consequences of small deviations from an ideal preparation. In this context one needs to distinguish the following two cases. First, there is a systematic error, such that the two "copies" may not match exactly, though the preparation of both first and second copies individually is reliable. And second, the preparation process could suffer from fluctuations from run to run, or, equivalently, it produces a mixed state.

A. Imperfect copies

Let us start with an investigation of the consequences that imperfect copies yield on the concurrence measurement. Instead of an ideal twofold copy $|\Psi\rangle \otimes |\Psi\rangle$, we assume that we are dealing with two different states $|\Psi\rangle \otimes |\Psi'\rangle$ with $|\langle\Psi|\Psi'\rangle| \lesssim 1$. Thus, we can express $|\Psi'\rangle$ as

$$|\Psi'\rangle = \sqrt{1-\epsilon^2}|\Psi\rangle + \epsilon|\delta\Psi\rangle, \quad (10)$$

with $\langle\Psi|\delta\Psi\rangle = 0$. The quantity c_m that one actually measures in a laboratory experiment reads

$$c_m = \sqrt{\langle\Psi| \otimes \langle\Psi'|A|\Psi\rangle \otimes |\Psi'\rangle}. \quad (11)$$

In contrast to the case of exact copies above, the value of c_m now depends on the choice of A; i.e., it depends on whether one measures $4(P_-^{(1)} \otimes P_-^{(2)})$, $4(P_-^{(1)} \otimes \mathbb{1}^{(2)})$, or $4(\mathbb{1}^{(1)} \otimes P_-^{(2)})$. However, the matrix elements of the kind $\langle\Psi| \otimes \langle\Psi|P_-^{(1)} \otimes P_+^{(2)}|\Psi\rangle \otimes |\delta\Psi\rangle$ vanish since $P_-^{(1)} \otimes P_+^{(2)}$ is globally antisymmetric, whereas $\langle\Psi| \otimes \langle\Psi|$ is globally symmetric. Therefore, the differences between the different choices for A are of order ϵ^2 and can be neglected in a first-order approximation.

In general, the states $|\Psi\rangle$ and $|\Psi'\rangle$ have a different value of concurrence, so before one can estimate the quality of the measurement of c, one needs to agree on what one wants to measure. Unless there is good reason to assume that the preparation of one "copy" is more precise than the other, we need to assume that both copies have been prepared with imperfection. Therefore, a good figure of merit is obtained by the comparison of c_m with the mean value $[c(\Psi)+c(\Psi')]/2$, which reads up to first order in ϵ

$$[c(\Psi)+c(\Psi')]/2 = c(\Psi) + \frac{\epsilon}{2c(\Psi)}[\langle\Psi| \otimes \langle\Psi|A|\Psi\rangle \otimes |\delta\Psi\rangle + \langle\delta\Psi| \otimes \langle\Psi|A|\Psi\rangle \otimes |\Psi\rangle]. \tag{12}$$

With the analogous expansion for c_m, one easily verifies that the difference of $[c(\Psi)+c(\Psi')]/2$ and c_m is of order ϵ^2, such that the present measurement scheme is insensitive to imperfect preparation of the copies, in first order.

Whereas there is no imperfection $|\delta\Psi\rangle$ that spoils the concurrence measurement in first order, typical errors have a detrimental impact of second, or higher order, with a weight which can depend on the choice of A. So, if one has an idea of what error there might be in the preparation process, one can compensate for that in some cases. For example, in the preparation in [8] it was guaranteed that the two states $|\Psi\rangle$ and $|\Psi'\rangle$ had the same Schmidt coefficients and their Schmidt bases coincided up to different local phases. In that case, one can compensate for the error completely by choosing A to be $4(P_-^{(1)} \otimes 1^{(2)})$ or $4(1^{(1)} \otimes P_-^{(2)})$. Then, indeed, one has $c(\Psi)=c(\Psi')=c_m$; i.e., the error vanishes exactly.

B. Mixed states

The second source of possible errors may result from an imperfect preparation procedure that produces mixed states instead of pure states [39]. A two-crystal type-I down-conversion source [18], with improper spatial mode matching and spectral filtering, may produce the state

$$\varrho = (1-\epsilon)|\Psi\rangle\langle\Psi| + \epsilon(|\alpha|^2|HH\rangle\langle HH| + |\beta|^2|VV\rangle\langle VV|), \tag{13}$$

instead of the ideal pure state $|\Psi\rangle = \alpha|HH\rangle + \beta|VV\rangle$. That is, the phase coherence between $|HH\rangle$ and $|VV\rangle$ is reduced by $1-\epsilon$ and ϱ is mixed. Consequently, the actual concurrence of ϱ is smaller than that of $|\Psi\rangle$ and reads [1]

$$c(\rho) = 2(1-\epsilon)|\alpha\beta|. \tag{14}$$

For the experimentally accessible quantities, however, one obtains

$$\sqrt{4\mathrm{Tr}[\varrho^{\otimes 2}(1^{(1)} \otimes P_-^{(2)})]} = \sqrt{4\mathrm{Tr}[\varrho^{\otimes 2}(P_-^{(1)} \otimes 1^{(2)})]} = 2|\alpha\beta|. \tag{15}$$

Thus, the relative error due to mixing is linear in ϵ.

If, however, one is not satisfied with this precision and aims at a reduction of the error, one can achieve an increase of precision by a factor of 2 if one measures not only on the two copies of one subsystem, but on both of the twin subsystems. The difference between the experimentally obtained value

$$\sqrt{4\mathrm{Tr}[\varrho^{\otimes 2}(P_-^{(1)} \otimes P_-^{(2)})]} = 2\sqrt{1-\epsilon-\epsilon^2/2}|\alpha\beta| \approx 2(1-\epsilon/2)|\alpha\beta| \tag{16}$$

and the exact one, given by Eq. (14), then amounts to only $\epsilon|\alpha\beta|$.

As a matter of fact, if one performs this second measurement, one does not only measure $P_-^{(1)} \otimes P_-^{(2)}$. Since P_- and P_+ add up to the identity, one also obtains $P_+^{(1)} \otimes P_-^{(2)}$ and $P_-^{(1)} \otimes P_+^{(2)}$ without any further experimental effort. Thus, one can employ the estimate of concurrence for mixed states [19],

$$c(\varrho) \geq \sqrt{\mathrm{Tr}(\varrho^{\otimes 2}V)} = 2(1-\epsilon)|\alpha\beta|, \tag{17}$$

with $V=4(P_-^{(1)} \otimes P_-^{(2)} - P_-^{(1)} \otimes P_+^{(2)})$, and, indeed, one obtains the exact result given by Eq. (14).

C. Imperfect copies of mixed states

It is not necessarily guaranteed that a preparation process suffers only from one of the above imperfections. Since it might be that one ends up preparing imperfect copies of mixed states, let us look at the consequences thereof for the measurement of concurrence. Suppose, for instance, that one of the copies is in a pure state ϱ given by Eq. (13), with $\epsilon=0$, while ϱ', the state of the second copy, has a finite value of ϵ. In that case the measurement of $A=4P_-^{(1)} \otimes P_-^{(2)}$ gives $c_m=2(1-\epsilon/4)|\alpha\beta|$; i.e., the experimentally obtained value lies between the two values of concurrence for the imperfect twin brothers—i.e., $c(\varrho) > c_m > c(\varrho')$. If on the other hand the bound of [19] is measured on two imperfect twins, then one obtains a valid bound for the product of the concurrences of the two states [40]:

$$c(\varrho)c(\varrho') \geq \sqrt{\mathrm{Tr}(\varrho \otimes \varrho' V)}. \tag{18}$$

That is, by assuming that one of the states was maximally entangled, one always finds a valid lower bound for the other one.

VI. MULTIPARTITE GENERALIZATIONS AND SCALING

The crucial figure of merit for the present approach is its scaling behavior with increasing system size: whereas a bipartite qubit system is well suited for academic investigations, only the applicability to large systems gives an honest estimate of the capabilities of the present measurement scheme.

A complete tomographic measurement of a quantum state of a $(d_1 \times d_2)$-dimensional system is comprised of $(d_1 d_2)^2 - 1$ individual observables. If, however, the state to be characterized is pure, concurrence can also be inferred by quantum-state tomography on only one subsystem, since it is a function of the reduced density matrix of a single subsystem [11]. In that case a tomographic measurement is comprised of d^2-1 observables, where d is the dimension of the smaller subsystem.

In the presently described approach, concurrence is inferred with a single observable. However, in laboratory experiments with high-dimensional quantum systems it might not always be practical to implement the projection onto the antisymmetric subspace in terms of a single measurement. Depending on the actual system one might need to perform projective measurements on individual antisymmetric states. Though even under the assumption of the worst case—that is, if one needs to decompose the projection onto P_- into projections onto one-dimensional subspaces—one has $d_i(d_i-1)/2$ $(i=1,2)$ measurements per subsystem to perform.

For a system of two three-level systems (qutrits) this amounts to 9 measurements as compared to 80 for quantum-state tomography on mixed states; for pure states, the present method requires 3 measurements, whereas tomography needs 8.

Also for multipartite generalization of concurrence [41] the present methods scales favorably as compared to quantum-state tomography. The exponential growth of global observables with increasing number of subsystems makes the characterization of entanglement properties of large states a very tedious task. Tomography on an n-partite system of qubits requires the determination of 4^n observables. With the present method, there are two observables to be measured per subsystem: the projections onto P_- and onto P_+. The number of *independent* n-partite system observables comprised of $P_\pm$ reads $2^n-1\approx 2^n$, that is the square root of the number of observables to be measured for quantum state tomography. For the largest system in which entangled states have been prepared so far [4], this means that instead of 65 535, only 256 observables would have to be measured in the present approach.

VII. CONCLUSION

In conclusion, we have shown that the concurrence of a pure state can be measured directly in various systems, provided one has access to two copies of the state. We have demonstrated this measurement experimentally using photons entangled in two degrees of freedom, which has allowed us to create the two copies on a single pair of entangled photons. Whereas state reconstruction and subsequent mathematical determination of entanglement is a viable and successfully demonstrated option for systems with few subsystems, more efficient approaches are required for larger objects. The strong appeal of our method is the association of a physical meaning to concurrence in terms of the results of a projective measurement on a twofold copy of the pure state. This was the first demonstration of the measurement of functions of the density matrix through projective measurements on multiple copies of a state. The proof-of-principle experiment described here suggests that it may be possible to avoid the large numbers of measurements required for state reconstruction and reliably characterize the entanglement properties directly, even in the unavoidable presence of experimental imperfections.

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