Method for Direct Measurement of the Wigner Function in Cavity QED and Ion Traps

L. G. Lutterbach and L. Davidovich
Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, 21945-970 Rio de Janeiro, RJ, Brasil
(Received 18 October 1996)

We show that the Wigner function corresponding to an electromagnetic field in a cavity or a vibronic state of a trapped ion can be directly measured by means of a simple experimental procedure. The proposed method is insensitive to the relatively low detection efficiency in cavity QED experiments and yields a direct physical meaning to the Wigner distribution. [S0031-9007(97)02790-7]

PACS numbers: 42.50.Dv, 03.65.Bz, 42.50.Ar, 42.50.Vk

Phase space representations are very useful in quantum mechanics since they allow the calculation of correlation functions of operators as classical-like integrals, and are also helpful for the study of the transition to classical physics. The oldest of such representations is due to Wigner [1], who used it as a convenient tool to calculate quantum corrections to classical statistical mechanics. It was shown by Moyal [2] that the quantum average of a Weyl-ordered (symmetric-ordered) function of the momentum and position operators could be expressed as a classical-like average of the corresponding classical function (in which the operators are replaced by c numbers), with the Wigner distribution acting as a weight measure in phase space [3]. The uncertainty principle forbids, however, the interpretation of this function as a probability distribution, since it is not possible to determine simultaneously the momentum and the position of a particle. In fact, it is easy to find examples of states for which the Wigner distribution assumes negative values. This fact may lead to the idea that it does not correspond to any directly measured quantity. Up until now, this notion has been upheld by the fact that the different schemes proposed so far to determine the Wigner distribution of a quantum system rely either on tomographic reconstructions from data obtained in homodyne measurements [4,5] or on convolutions obtained by photon counting [6]. It is the purpose of this Letter to show that the Wigner function can be directly measured, through a very simple scheme, especially suitable to experiments in cavity quantum electrodynamics and in ion traps. This is especially important in view of recent experimental results concerning the production and detection of coherent superpositions of localized mesoscopic states [7,8]. In these experiments, the existence of coherence is inferred from partial information obtained about the system. A method yielding more complete knowledge onf the quantum states involved would be highly desirable. Quantum tomography schemes for determining the vibrational state of a trapped ion were proposed in [5]. In cavity QED, information on the field must be obtained from probe atoms, since the high Q of the cavity and the weak intensities involved do not allow the direct measurement of the field. A method for realizing the "quantum endoscopy" of a field in a cavity, through a beam of probe atoms, was presented in [9]. While it holds only for pure states, an analog procedure for trapped ions leads to the quantum endoscopy of the vibrational state, even for mixed states [10]. Other proposals for measuring the state of a field in a resonator were discussed in [11]. More recently, a beautiful experiment led to the determination of the density matrices and Wigner functions of various quantum states of motion of a trapped ion [12].

The Wigner function belongs to a general class of phase-space distributions, which for a one-dimensional harmonic oscillator with annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ can be written as [13]

$$W(\alpha, s) = \mathrm{Tr}[\hat{\rho}\hat{T}(\alpha, s)], \qquad (1)$$

where

$$\hat{T}(\alpha, s) = \int e^{\alpha\xi^* - \alpha^*\xi} e^{s|\xi|^2/2} \hat{D}(\xi) \pi^{-1} d^2\xi, \qquad (2)$$

and $\hat{D}(\xi) = \exp(\xi\hat{a}^\dagger - \xi^*\hat{a})$ is the displacement operator, with ξ being a complex number. The operator $\hat{D}(\xi)$ represents the action of a classical current on the field: It yields the coherent state $|\xi\rangle$ [14] when applied to the harmonic oscillator ground state. Note that $W(\alpha, s)$ is real when s is real. For $s = 0$, one obtains the Wigner distribution, while $s = -1$ and $s = 1$ correspond, respectively, to the Q and the Glauber-Sudarshan P representations (which allow the calculation of quantum averages of operators in antinormal and normal order, respectively [3,13,14]). An alternative expression for $\hat{T}(\alpha, s)$, very useful for our purposes, is [13]

$$\hat{T}(\alpha, s) = \frac{2}{1 - s} \hat{D}(\alpha) \left[\frac{s + 1}{s - 1}\right]^{\hat{a}^\dagger\hat{a}} \hat{D}^{-1}(\alpha). \qquad (3)$$

Setting $s = -i \cot\phi$, (1) and (3) become

$$W(\alpha, \phi) = -2ie^{i\phi} \sin\phi \, \mathrm{Tr}[\hat{D}(-\alpha)\hat{\rho}\hat{D}(\alpha)e^{2i\phi\hat{a}^\dagger\hat{a}}]. \qquad (4)$$

For $\phi = \pi/2$ ($s = 0$), we get the Wigner distribution [13]

$$W(\alpha) = 2\,\mathrm{Tr}[\hat{\rho}\hat{D}(\alpha)e^{i\pi\hat{a}^\dagger\hat{a}}\hat{D}^{-1}(\alpha)]. \qquad (5)$$

The Q and P representations correspond to imaginary values of ϕ in (4).

Let us consider first the measurement of the field inside a cavity, in a typical cavity QED experiment [15] (see Fig. 1). A high-Q superconducting cavity C, containing the field to be measured (in the microwave region), is placed between two low-Q cavities (R_1 and R_2 in Fig. 1). The cavities R_1 and R_2 are connected to the same microwave generator, and a dephaser between this generator and R_2 allows one to change the relative phase η between the fields in R_1 and R_2. Another microwave source is connected to C, allowing the injection of a coherent state in the empty cavity or the displacement of an already existing field, through the operator $\hat{D}(\alpha)$. This system is crossed by a velocity-selected atomic beam, such that an atomic transition $e \leftrightarrow g$ is resonant with the fields in R_1 and R_2, and quasiresonant (detuning δ) with the field in C. Just before R_1, the atoms are promoted to the highly excited circular Rydberg state $|e\rangle$. As each atom crosses the low-Q cavities, it sees a $\pi/2$ pulse, so that $|e\rangle \to [|e\rangle + e^{i\eta}|g\rangle]/\sqrt{2}$, and $|g\rangle \to [-e^{-i\eta}|e\rangle + |g\rangle]/\sqrt{2}$, with η replaced by 0 in R_1. The atom interacts dispersively with the field in cavity C, so that transitions $e \leftrightarrow g$ can be neglected, but there is a state-dependent energy shift of the atom-field system (Stark shift), which dephases the field, after an interaction time $t_{\rm int}$ between the atom and the cavity mode. The dephasings corresponding to the states $|e\rangle$ and $|g\rangle$ are implemented, respectively, by the unitary operators $\hat{\mathcal{T}}_e(\phi) = \exp[i\phi(\hat{a}^\dagger\hat{a} + 1)]$ and $\hat{\mathcal{T}}_g(\phi) = \exp(-i\phi\hat{a}^\dagger\hat{a})$, where $\phi = (\Omega^2/\delta)t_{\rm int}$ is the one-photon phase shift, and the Rabi frequency Ω measures the coupling between the atom and the cavity mode.

The direct measurement of the Wigner function of the field in C involves the following steps. One turns on the microwave source connected to C for some time Δt, so that the field to be measured gets displaced, its density operator $\hat{\rho}$ being replaced by $\hat{\rho}' = \hat{D}(\alpha)\hat{\rho}\hat{D}^{-1}(\alpha)$. A probe atom in state $|e\rangle$ is then sent through the system. After the atom crosses R_1, C, and R_2, the entangled atom-field state $\hat{\rho}_{\rm atom+field}$ becomes

$$\begin{aligned}\tfrac{1}{4}[&|e\rangle\langle e| \otimes (\hat{\mathcal{T}}_e - e^{-i\eta}\hat{\mathcal{T}}_g)\hat{\rho}'(\hat{\mathcal{T}}_e^\dagger - e^{i\eta}\hat{\mathcal{T}}_g^\dagger) \\ &+ |g\rangle\langle g| \otimes (\hat{\mathcal{T}}_g + e^{i\eta}\hat{\mathcal{T}}_e)\hat{\rho}'(\hat{\mathcal{T}}_g^\dagger + e^{-i\eta}\hat{\mathcal{T}}_e^\dagger) \\ &+ \text{terms nondiagonal in atomic space}]. \end{aligned} \tag{6}$$

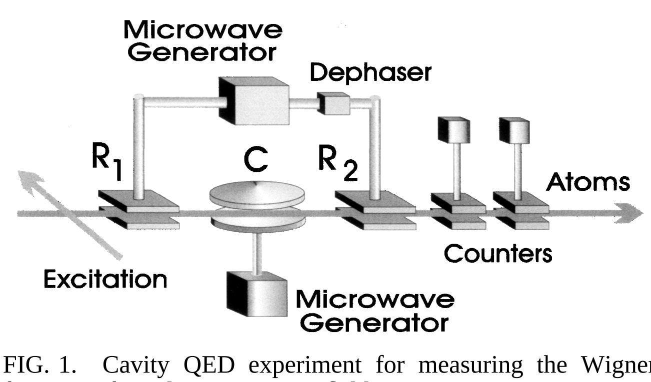

FIG. 1. Cavity QED experiment for measuring the Wigner function of an electromagnetic field in cavity C.

Finally, the internal state of the atom is detected by two field ionization detectors (see Fig. 1). This experiment is repeated many times, starting each run with the same field, and the probabilities P_e and P_g of detecting the probe atom in states e or g are determined. It is easy to show that

$$\Delta P = -\mathrm{Re}\{e^{i(\eta+\phi)}\,\mathrm{Tr}[\hat{D}(\alpha)\hat{\rho}\hat{D}^{-1}(\alpha)e^{2i\phi\hat{a}^\dagger\hat{a}}]\}, \tag{7}$$

where $\Delta P = P_e - P_g$. If $\phi = \pi/2$, and the dephasing is chosen so that $\eta = \pi/2$, this yields

$$\Delta P = \mathrm{Tr}[\hat{D}(\alpha)\hat{\rho}\hat{D}^{-1}(\alpha)e^{i\pi\hat{a}^\dagger\hat{a}}]. \tag{8}$$

Comparing this expression with Eq. (5), we arrive at the very simple relation

$$\Delta P = P_e - P_g = W(-\alpha)/2. \tag{9}$$

The difference between the two probabilities is therefore a direct measurement of the Wigner function of the field inside the cavity. One should note that this method allows one to choose the region of phase space to be explored. This is an especially useful feature, since one has frequently an idea of the region in which the Wigner function should be concentrated. From (9), one sees immediately that the Wigner function must be bounded between $+2$ and -2, a well-known mathematical property of W [13], which is given here a physical meaning. Also, the Wigner function attains negative values whenever $P_g > P_e$.

If the phase shift is different from $\pi/2$, one can see from (7) that by changing η one may detect the real and the imaginary part of $W(\alpha, \phi)$ given by (4). Therefore, one can measure phase space representations corresponding to imaginary values of s. The connection between $W(\alpha, \phi)$ and $W(\alpha) \equiv W(\alpha, \pi/2)$ can be obtained in the following way. It is easy to show from (1) and (2) that, setting $\tau = is$ and $\alpha = x + iy$,

$$i\,\frac{\partial W(x,y,\tau)}{\partial \tau} = -\frac{1}{8}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)W(x,y,\tau), \tag{10}$$

so that $W(x,y,\tau)$ obeys a free-particle Schrödinger equation, the parameter τ playing the role of a time [16]. As ϕ changes from $\pi/2$ to 0, and τ changes correspondingly from $\tau = 0$ to $\tau = \infty$, the behavior of $W(\alpha, \phi)$, illustrated in Fig. 2, is easily understandable in terms of the development in time of a free wave packet. In particular, the vanishing of $W(\alpha, \phi)$ when $\phi = 0$ is a direct consequence of the wave packet spreading. The interference fringes at the origin, displayed in Fig. 2(a), may be thought of as resulting from the collision of the two wave packets which, in Fig. 2(b), are shown flying apart from the collision region (origin of the phase space). Equation (10) also implies that $W(\alpha, \tau)$ is connected to $W(\alpha)$ through the free-particle propagator.

An important feature of our scheme is the insensitivity to the detection efficiency of the atomic counters [of the order of $(40 \pm 15)\%$ in recent experiments [8]], as long as $|e\rangle$ and $|g\rangle$ are detected with the same efficiency

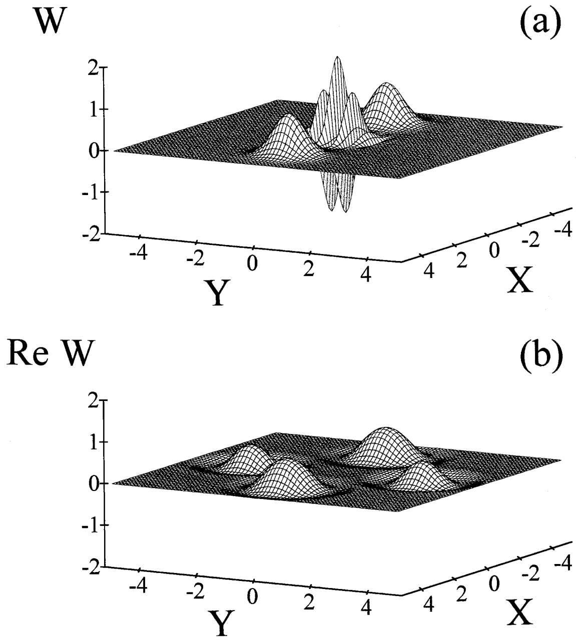

FIG. 2. Generalized phase-space distribution for the state $(|\alpha_0\rangle + |-\alpha_0\rangle)/\sqrt{2}$, with $|\alpha_0|^2 = 9$. The change of the one-photon phase shift ϕ from $\pi/2$ to 0 is equivalent to the time evolution of a wave packet in phase space from $\tau = 0$ to $\tau = \infty$. (a) $\phi = \pi/2$, corresponding to the (real) Wigner distribution (initial wave packet); (b) real part of $W(\alpha, \phi = \pi/4)$.

[17]. Indeed, if an atom is not detected after interacting with the cavity mode, the next atom will find a field described by the reduced density operator obtained from (6) by tracing out the atomic states: $\hat{\rho}'' = \frac{1}{2}(\hat{\mathcal{T}}_g \hat{\rho}' \hat{\mathcal{T}}_g^\dagger + \hat{\mathcal{T}}_e \hat{\rho}' \hat{\mathcal{T}}_e^\dagger)$. The value of ΔP for this second atom is then

$$\Delta P = -\frac{1}{2}\,\mathrm{Re}\{e^{i\eta}\,\mathrm{Tr}[(\hat{\mathcal{T}}_g \hat{\rho}' \hat{\mathcal{T}}_g^\dagger + \hat{\mathcal{T}}_e \hat{\rho}' \hat{\mathcal{T}}_e^\dagger)\hat{\mathcal{T}}_g^\dagger \hat{\mathcal{T}}_e]\}, \tag{11}$$

which reduces to (7), since $[\hat{\mathcal{T}}_g, \hat{\mathcal{T}}_e] = 0$.

The measurement accuracy does depend, however, on the detector's selectivity, that is, the ability to distinguish between the two atomic states. Another possible source of error is the velocity spread of the atomic beam, which would produce an uncertainty in the angle ϕ and in the angles of rotation in R_1 and R_2. For a 1% velocity spread and for average photon numbers of the order of 10, our calculations show that the distortion in ΔP is at most equal to 0.05 (the corresponding distortion in W is at most equal to 0.1), in the relevant region of phase space, so that the measured distribution is practically undistinguishable from the true one. In fact, the insensitivity of the proposed scheme to the detection efficiency allows a passive selection of atomic velocity (only the atom which goes through the detectors at the right time after excitation is detected).

We show now how a similar procedure could be applied to measure directly the Wigner function of the vibrational state of a trapped ion. The relevant level scheme is shown in Fig. 3. States $|\downarrow\rangle$ and $|\uparrow\rangle$ correspond to two metastable ground-state hyperfine sublevels ($^2S_{1/2}$, with $F = 2$, $m_F = -2$, and $F = 1$, $m_F = -1$, respectively), separated by $\hbar\omega_{\mathrm{HF}}$. The ion is trapped in a harmonic potential, and the vibrational levels associated with each electronic state $|\downarrow\rangle$ and $|\uparrow\rangle$ are also sketched in Fig. 3. We consider for simplicity a one-dimensional center-of-mass motion, with frequency ω, which can actually be realized by proper alignment of the laser beams [7]. States $|\downarrow\rangle$ and $|\uparrow\rangle$ are coupled by a stimulated Raman transition via two optical fields A and B (wave vectors $\vec{k}_A$ and $\vec{k}_B$), detuned by Δ from the electric dipole transitions (coupling strengths g_1 and g_2) from $|\downarrow\rangle$ and $|\uparrow\rangle$ to a third level a ($^2P_{1/2}$, $F = 2$, $m_F = -2$). When the frequency difference between these two beams is tuned near ω_{HF} ("carrier beams"), there are Rabi oscillations between the internal states $|\uparrow\rangle$ and $|\downarrow\rangle$. The Rabi frequency depends, however, on the vibrational state. For a Fock state $|n\rangle$, it is given by [18,19] $\Omega_n = \Omega_0[1 - (n + \frac{1}{2})\eta^2 + \frac{1}{4}(n^2 + n + \frac{1}{2})\eta^4 + O(\eta^6)]$, where the Lamb-Dicke parameter η is the square root of the ratio between the recoil energy and the vibrational quantum of energy $\hbar\omega$, and $\Omega_0 = 2g_1g_2/\Delta$. Typically, η is of the order of $0.1 - 0.2$, so that the term proportional to η^4 can be neglected if the relevant n's are sufficiently small (for $\eta = 0.1$, one should have $n < 10$ for an error smaller than 2.5%). On the other hand, application of the "displacement beams" B and C, with a frequency difference near ω, is formally equivalent to applying the displacement operator to the state of motion. Beam C is circularly polarized (σ_-), and does not couple $|\downarrow\rangle$ to any virtual $^2P_{1/2}$ state, so that only the motional state correlated with $|\uparrow\rangle$ is displaced. A fourth level b is used for detecting the electronic state of the ion (and also for Doppler precooling): A pulse D resonant with the $|\downarrow\rangle \to |b\rangle$ transition leads to a fluorescence signal if the ion is in $|\downarrow\rangle$, while the absence of fluorescence implies that the ion is in $|\uparrow\rangle$ (for the experiment reported in [7], the detection efficiency is close to 100%). We assume that the initial state of the system is $\hat{\rho}_{\mathrm{system}} = \hat{\rho}_v \otimes |\uparrow\rangle\langle\uparrow|$, where ρ_v is the density operator for the center-of-mass motional state. The Wigner

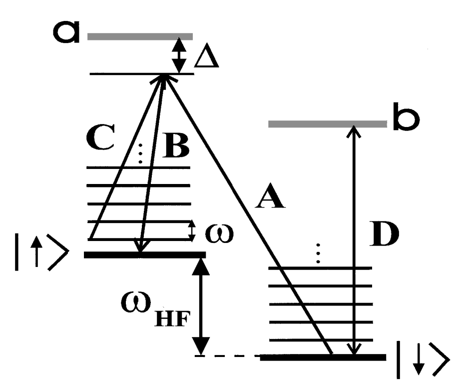

FIG. 3. Trapped ion: relevant level scheme (not in scale).

function of the field state correlated with $|\uparrow\rangle$ is measured in the following way. One applies first the displacement fields, so that $\hat{\rho}_v \rightarrow \hat{\rho}_v' = \hat{D}(\alpha)\hat{\rho}_v\hat{D}^{-1}(\alpha)$. The next procedure corresponds to the first step of a recently proposed quantum nondemolition scheme for measuring the population of the vibrational levels [18,19]. The carrier fields A and B are applied, with a time duration τ such that $\theta(\tau) \equiv \Omega_0\eta^2\tau/2 = \pi/2$ (for the parameters in [7], one should have $\tau \approx 50\,\mu$s). This implies that $\Omega_n\tau/2 = \Phi - (\pi/2)n$, where $\Phi = (\Omega_0\tau/2) - (\pi/4)$. The state of the system becomes

$$[|\uparrow\rangle \cos(\Phi - \pi\hat{a}^\dagger\hat{a}/2) + |\downarrow\rangle \sin(\Phi - \pi\hat{a}^\dagger\hat{a}/2)]\hat{\rho}_v' \times [\langle\uparrow| \cos(\Phi - \pi\hat{a}^\dagger\hat{a}/2) + \langle\downarrow| \sin(\Phi - \pi\hat{a}^\dagger\hat{a}/2)]. \quad (12)$$

The probabilities of finding the ion in $|\uparrow\rangle$ or $|\downarrow\rangle$ satisfy

$$P_\uparrow - P_\downarrow = \cos(2\Phi)\,\mathrm{Tr}[\hat{D}(\alpha)\hat{\rho}\hat{D}^{-1}(\alpha)e^{i\pi\hat{a}^\dagger\hat{a}}], \quad (13)$$

and therefore $P_\uparrow - P_\downarrow = \cos(2\Phi)W(-\alpha)/2$, yielding directly the Wigner function. A proper choice of α leads to the value of the Wigner function at any point of the phase space. One should note that a very precise calibration of both the amplitude and the phase of the displacement has been achieved in recent experiments [7].

It is interesting to compare our method with the procedure described in Ref. [12]. There a coherent displacement α is also applied to the input motional state. A resonant exchange between states $|\downarrow\rangle|n\rangle$ and $|\uparrow\rangle|n+1\rangle$ is then induced for a time t. For each time t and each α the population $P_\downarrow(t,\alpha)$ of the state $|\downarrow\rangle$ is measured. If $|\downarrow\rangle$ is the internal state at $t = 0$, the signal averaged over many measurements is $P_\downarrow(t,\alpha) = \frac{1}{2}[1 + \sum_{n=0}^\infty Q_n(\alpha)\cos(2\Omega_{n,n+1}t)e^{-\gamma_n t}]$, where $\Omega_{n,n+1}$ are the Rabi frequencies, γ_n the corresponding (experimentally determined) decay constants, and $Q_n(\alpha) = \langle n|\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle$ is the population distribution of the displaced state. The dependence of $\Omega_{n,n+1}$ on n allows the determination of $Q_n(\alpha)$ from $P_\downarrow(t,\alpha)$ [12]. On the other hand, from $Q_n(\alpha)$ one can calculate the Wigner function, as may be seen by evaluating the trace with respect to the number-state basis in (5), which leads to $W(\alpha) = 2\sum_{n=0}^\infty(-1)^n Q_n(\alpha)$. In our method, the induction of Rabi oscillations between $|\uparrow\rangle$ and $|\downarrow\rangle$ by means of carrier beams can be thought of as leading to the experimental determination of the sum $\sum_{n=0}^\infty(-1)^n Q_n(\alpha)$, and therefore to a direct measurement of the Wigner function, at any time t. The fact that measurements over time intervals are not necessary here implies that our method is especially useful for systems in which decoherence plays an important role.

In conclusion, we have shown that it is possible to measure directly the Wigner function of the electromagnetic field in a cavity, or the vibrational state of an ion in a trap. Our method can be applied to recent experiments involving the production of "Schrödinger-cat" states of the field in a cavity or of the center of mass of a trapped ion, leading to the direct probing of the value of the Wigner function at any point of phase space.

We thank C. Monroe for useful remarks. This work was partially supported by the Brazilian agencies Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), and Coordenação de Aperfeiçoamento de Pessoal de Ensino Superior (CAPES).

[1] E. Wigner, Phys. Rev. **40**, 749 (1932).
[2] J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).
[3] For a review, see M. Hillery *et al.*, Phys. Rep. **106**, 121 (1984).
[4] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989); D. T. Smithey *et al.*, Phys. Rev. Lett. **70**, 1244 (1993); U. Leonhardt and H. Paul, Phys. Rev. Lett. **72**, 4086 (1994); G. M. D'Ariano *et al.*, Phys. Rev. A **52**, R1801 (1995); T. J. Dunn *et al.*, Phys. Rev. Lett. **74**, 884 (1995).
[5] S. Wallentowitz and W. Vogel, Phys. Rev. Lett. **75**, 2932 (1995); C. D'Helon and G. Milburn, Phys. Rev. A **54**, R25 (1996); J. F. Poyatos *et al.*, Phys. Rev. A **53**, R1966 (1996).
[6] K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **76**, 4344 (1996); S. Wallentowitz and W. Vogel, Phys. Rev. A **53**, 4528 (1996).
[7] C. Monroe *et al.*, Science **272**, 1131 (1996).
[8] M. Brune *et al.*, Phys. Rev. Lett. **77**, 4887 (1996).
[9] P. J. Bardroff *et al.*, Phys. Rev. A **51**, 4963 (1995); **53**, 2736 (1996).
[10] P. J. Bardroff *et al.*, Phys. Rev. Lett. **77**, 2198 (1996).
[11] M. Wilkens and P. Meystre, Phys. Rev. A **43**, 3832 (1991); M. Freyberger and A. M. Herkommer, Phys. Rev. Lett. **72**, 1952 (1994); S. M. Dutra and P. L. Knight, Phys. Rev. A **49**, 1506 (1994).
[12] D. Leibfried *et al.*, Phys. Rev. Lett. **77**, 4281 (1996).
[13] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).
[14] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[15] L. Davidovich *et al.*, Phys. Rev. Lett. **71**, 2360 (1993); Phys. Rev. A **53**, 1295 (1996).
[16] This equation corresponds to the diffusion equation found in [13], and it is obtained from it when the parameter s becomes purely imaginary.
[17] We note that the relative efficiency between the atomic detectors can be determined experimentally with high accuracy.
[18] R. L. de Matos Filho and W. Vogel, Phys. Rev. Lett. **76**, 4520 (1996).
[19] L. Davidovich *et al.*, Phys. Rev. A **54**, 5118 (1996).