# Speed limits in general relativity

Robert J Low†

Mathematics Division, School of Mathematical and Information Sciences, Coventry University, Priory Street, Coventry CV1 5FB, UK

Received 10 September 1998, in final form 27 November 1998

**Abstract.** Some standard results on the initial value problem of general relativity in matter are reviewed. These results are applied first to show that in a well defined sense, finite perturbations in the gravitational field travel no faster than light, and second to show that it is impossible to construct a warp drive as considered by Alcubierre (1994 The warp drive: hyper-fast travel within general relativity *Class. Quantum Grav.* **11** L73–7) in the absence of exotic matter.

PACS numbers: 0420E, 0420G

## 1. Introduction

This paper is divided into four sections. In the first, I review some standard results on the initial value problem of general relativity, both for vacuum spacetimes and for spacetimes containing matter. In the second section, I consider the propagation of gravitational disturbances. It is well known that in the limit of small perturbations, such disturbances travel along null geodesics, and so gravitational waves travel at light speed. An initial value approach to non-infinitesimal perturbations will be developed that shows that, in a well-defined sense, finite changes in the gravitational field propagate no faster than light speed. In the third section, I consider the problem of whether it is possible to construct a 'warp drive', such as is described in Alcubierre (1994). This necessitates some way of describing what it means to do such a thing in a deterministic theory; after developing such a notion I show that it is impossible to build a warp drive if all matter present satisfies the dominant energy condition. Alcubierre's conjecture that exotic matter is required for the construction of a warp drive is therefore seen to be correct. Finally, I give a brief discussion of the results.

## 2. The initial value problem

In this section I will describe some standard results. Outlines of the proofs may be found in Chapter 10 of Wald (1984); detailed proofs are in Chapter 7 of Hawking and Ellis (1973), and a more geometric approach which sets up a Hamiltonian formulation for the vacuum case is presented in Fischer and Marsden (1976).

Let $(M, g_{\alpha\beta})$ be a Lorentz manifold satisfying the Einstein equations, so that $G_{\alpha\beta} = 0$. Then if $\Sigma$ is a spacelike surface in $M$, the metric $g_{\alpha\beta}$ induces on $\Sigma$ a Riemannian 3-metric, $h_{ab}$,

† E-mail address: r.low@coventry.ac.uk

0264-9381/99/020543+07$19.50  

and a symmetric tensor $K_{ab}$, the second fundamental form of $\Sigma$. The following constraints are implied by the Gauss–Codazzi relations:

$$\begin{aligned} &D_b K^b{}_a - D_a K^b{}_b = 0 \\ &{}^{(3)}R + (K^a{}_a)^2 - K_{ab}K^{ab} = 0 \end{aligned} \tag{2.1}$$

where $D_a$ is the covariant derivative on $\Sigma$ defined by $h_{ab}$, and ${}^{(3)}R$ is the scalar curvature of $h$.

Define a vacuum initial data set to be a triple $(\Sigma, h_{ab}, K_{ab})$ where $\Sigma$ is a smooth 3-manifold, $h_{ab}$ is a Riemannian metric on $\Sigma$, and $K_{ab}$ is a symmetric tensor on $\Sigma$ satisfying the constraint equations (2.1). Then we have the following theorem:

**Theorem 2.1.** *Let* $(\Sigma, h_{ab}, K_{ab})$ *be a vacuum initial data set. Then*

*(i) There exists a unique smooth spacetime* $(M, g_{\alpha\beta})$ *known as the maximal development of* $(\Sigma, h_{ab}, K_{ab})$ *and a smooth embedding* $i : \Sigma \to M$ *such that* $M$ *is globally hyperbolic with Cauchy surface* $i(M)$*, the metric and second fundamental form induced on* $i(\Sigma)$ *by* $g_{\alpha\beta}$ *are, respectively,* $i_*(g_{ab})$ *and* $i_*(K_{ab})$*. Furthermore, any other spacetime satisfying these conditions may be mapped isometrically into* $(M, g_{\alpha\beta})$*.*
*(ii) If* $(\Sigma', h'_{ab}, K'_{ab})$ *is a vacuum initial data set with maximal development* $(M', g'_{\alpha\beta})$ *and embedding* $i' : \Sigma' \to M'$*, and there is a diffeomorphism from* $S \subseteq \Sigma$ *to* $S' \subseteq \Sigma'$ *taking* $(h_{ab}, K_{ab})$ *on* $S$ *to* $(h'_{ab}, K'_{ab})$ *on* $S'$*, then the Cauchy development* $D(i(S))$ *in* $M$ *is isometric to the Cauchy development* $D(i'(S'))$ *in* $M'$*.*
*(iii) The metric* $g_{\alpha\beta}$ *on* $M$ *depends continuously on the vacuum initial data* $(h_{ab}, K_{ab})$ *on* $\Sigma$*.*

It is worth noting that $M$ need not be an inextensible spacetime; however, if $j : M \to M'$ is an isometry onto its image and fails to be surjective, $j(i(\Sigma))$ cannot be a Cauchy surface for $M'$. Fischer (1995) has an interesting discussion of this and related issues.

Of more relevance here, however, is the following simple consequence, obtained by letting $S = \Sigma$ in part (ii) of the above theorem.

**Corollary 2.2.** *Let* $(\Sigma, h_{ab}, K_{ab})$ *and* $(\Sigma', h'_{ab}, K'_{ab})$ *be vacuum initial data sets such with maximal developments* $(M, g_{\alpha\beta})$ *and* $M', g'_{\alpha\beta})$ *respectively, and suppose that there is a diffeomorphism* $j$ *mapping* $\Sigma$ *to a subset of* $\Sigma'$ *which takes* $(h_{ab}, K_{ab})$ *to* $(h'_{ab}, K'_{ab})$*. Then* $M$ *is isometric to* $D(i'(j(\Sigma)))$ *in* $M'$*.*

Similar results hold for the Einstein equations with certain forms of matter present.

In particular, suppose we have matter described by a vector $\boldsymbol{u}$ satisfying a quasi-linear symmetric hyperbolic system, i.e. a differential equation of the form

$$A^\alpha \nabla_\alpha \boldsymbol{u} + B\boldsymbol{u} = \boldsymbol{c} \tag{2.2}$$

where

(i) $A^\alpha$ is a 4-vector of symmetric matrices,
(ii) $B$ is a matrix, and
(iii) $\boldsymbol{c}$ is a vector

and $A^\alpha$, $B$ and $\boldsymbol{c}$ can all depend on $\boldsymbol{u}$ as well as position. We will mostly be interested in the case where the ray cone for the system (the boundary of the set of vectors $v^\alpha$ making $v_\alpha A^\alpha$ positive definite) lies inside the light cone.

It is a standard result (Courant and Hilbert 1962) that this includes the case of matter described by a second-order hyperbolic system where the bicharacteristics are timelike or null. The physical significance of this constraint on $A^\alpha$ is that the bicharacteristics of the matter equations are causal curves, so that the matter cannot support the transfer of information at a speed exceeding that of light.

Suppose also that the stress tensor of the matter present, $T_{\alpha\beta}$, depends only on $\boldsymbol{u}$ and $g^{\alpha\beta}$.

We will be concerned with the issue of whether it is possible to construct a warp drive in the absence of exotic matter; to this end, we require a definition of exotic matter;

**Definition 2.3.** A matter field will be called *physically reasonable* if it satisfies a symmetric hyperbolic evolution equation and its stress-energy tensor satisfies the dominant energy condition; otherwise it will be called *exotic*. □

It now follows that if $\Sigma$ is a spacelike surface in $(M, g_{\alpha\beta})$ where $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, then again denoting the induced Riemannian metric on $\Sigma$ by $h_{ab}$ and its second fundamental form by $K_{ab}$, we find

$$\begin{aligned} &D_b K^b{}_a - D_a K^b{}_b = -8\pi J_b \\ &{}^{(3)}R + (K^a{}_a)^2 - K_{ab}K^{ab} = 16\pi\rho \end{aligned} \tag{2.3}$$

where $\rho = T_{\alpha\beta}n^\alpha n^\beta$ and $J_a$ is the projection to $\Sigma$ of $T_{\alpha\beta}n^\beta$, $n^\alpha$ being the future pointing normal to $\Sigma$.

We can now define an initial data set $(\Sigma, h_{ab}, K_{ab}, J_a, \rho)$ as a 3-manifold with a 3-metric $h_{ab}$, a symmetric tensor $K_{ab}$, a vector $J_a$ and a scalar $\rho$ satisfying the above equations. There follows a theorem analogous to theorem 2.1 above, except that now the metric $g_{\alpha\beta}$ satisfies $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ where $T_{\alpha\beta}$ is the stress tensor of the matter field described by $u$ and satisfying the evolution equation (2.2). Similarly, there is an analogous corollary to corollary 2.2.

Explicitly, we have

**Theorem 2.4.** *Let $(\Sigma, h_{ab}, K_{ab}, J_a, \rho)$ be an initial data set. Then*

*(i) There exists a unique smooth spacetime $(M, g_{\alpha\beta})$ known as the maximal development of $(\Sigma, h_{ab}, K_{ab}, J_a, \rho)$ and a smooth embedding $i : \Sigma \to M$ such that $M$ is globally hyperbolic with Cauchy surface $i(M)$, the metric and second fundamental form induced on $i(\Sigma)$ by $g_{\alpha\beta}$ are, respectively, $i_*(g_{ab})$ and $i_*(K_{ab})$, and $J_a$ and $\rho$ are induced on $\Sigma$ by a matter field satisfying (2.2). Furthermore, any other spacetime satisfying these conditions may be mapped isometrically in $(M, g_{\alpha\beta})$.*

*(ii) If $(\Sigma', h'_{ab}, K'_{ab}, J'_a, \rho')$ is a vacuum initial data set with maximal development $(M', g'_{\alpha\beta})$ and embedding $i' : \Sigma' \to M'$, and there is a diffeomorphism from $S \subseteq \Sigma$ to $S' \subseteq \Sigma'$ taking $(h_{ab}, K_{ab}, J_a, \rho)$ on $S$ to $(h'_{ab}, K'_{ab}, J'_a, \rho')$ on $S'$, then the Cauchy development $D(i(S))$ in $M$ is isometric to the Cauchy development $D(i'(S'))$ in $M'$.*

*(iii) The metric $g_{\alpha\beta}$ on $M$ depends continuously on the initial data $(h_{ab}, K_{ab}, J_a, \rho)$ on $\Sigma$.*

with corollary

**Corollary 2.5.** *Let $(\Sigma, h_{ab}, K_{ab}, J_a, \rho)$ and $(\Sigma', h'_{ab}, K'_{ab}, J'_a, \rho')$ be initial data sets such with maximal developments $(M, g_{\alpha\beta})$ and $(M', g'_{\alpha\beta})$ respectively, and suppose that there is a diffeomorphism $j$ mapping $\Sigma$ to a subset of $\Sigma'$ which takes $(h_{ab}, K_{ab}, J_a, \rho)$ to $(h'_{ab}, K'_{ab}, J'_a, \rho')$. Then $M$ is isometric to $D(i'(j(\Sigma)))$ in $M'$.*

It is perhaps worth noting here that this theorem and corollary are true for a *fixed* equation of the form (2.2). One could have identical initial conditions with non-isometric maximal developments if the physical properties of the matter fields involved, and hence the evolution equations, were different. In the following, we will consider only the effects of changing the initial conditions, while keeping the evolution equation fixed.

## 3. Evolution of gravitational perturbations

It is well known that linear perturbations of the metric on an Einstein vacuum satisfy a wave equation, and that the bicharacteristics of this wave equation are the null geodesics of the background spacetime (Wald 1984). In this sense, then, small perturbations in the gravitational field propagate at the speed of light.

However, it is somewhat more problematic to describe what happens with a finite perturbation of the metric. The difficulty is two-fold. First, there is the problem of deciding just what the perturbation is—the Einstein equations are non-linear, so there is no natural splitting into background field and disturbance. Second, since the metric itself determines the null geodesics, what can it mean to say that perturbations in the metric travel at light speed?

Let us approach this by first recasting the case of infinitesimal perturbations. So, let $\Sigma$ be a Cauchy surface for $(M, g_{\alpha\beta})$, and let $K$ be a submanifold of $\Sigma$. If we make an infinitesimal change to the vacuum initial data on $K$, while leaving that on $\Sigma \setminus K$ unchanged, then the metric is unchanged on $D(\Sigma \setminus K)$, since the perturbation travels at light speed and therefore cannot reach any point outside $I(K)$.

So now consider the effect of changing the initial data on $K$ by a finite amount. We can no longer split the spacetime into a background metric with a perturbation, but must proceed as follows: denote the first vacuum initial data set by $(\Sigma, h_{ab}, K_{ab})$, and the altered one by $(\Sigma, h'_{ab}, K'_{ab})$, noting that $(h_{ab}, K_{ab})$ and $(h'_{ab}, K'_{ab})$ agree outside $K$. Also, denote the first maximal development by $(M, g_{\alpha\beta})$ and the other by $(M', g'_{\alpha\beta})$.

By restricting we obtain an initial data set $(\Sigma \setminus K, h_{ab}, K_{ab})$, with associated maximal development $(M^0, g^0_{\alpha\beta})$. It follows immediately from corollary 2.2 that there is an isometry from $M^0$ to $D(i(\Sigma \setminus K))$ in $M$, and also to $D(i'(\Sigma \setminus K))$ in $M'$. We thus obtain

**Corollary 3.1.** *With all quantities as defined above, there is an isometry from* $D(i(\Sigma \setminus K)) \subset M$ *to* $D(i'(\Sigma \setminus K)) \subset M'$.

Since in $M$ no influence can propagate from a point of $K$ to a point of $D(i(\Sigma \setminus K))$ along a causal curve, and $M'$ differs from $M$ only outside $D(i(\Sigma \setminus K)) \equiv D(i'(\Sigma \setminus K))$, we can interpret this result as stating that no gravitational influence propagates faster than light.

## 4. Warp drive

Let us now proceed to the somewhat more subtle question of whether it is possible to construct a warp drive. Analysis (Pfenning and Ford 1997) has shown that the particular metric considered by Alcubierre (1994) is unphysical; there remains the question of whether there is a physically reasonable metric exhibiting analogous properties.

Before attempting to answer the question, it is necessary to find a way of asking it. The difficulty is that in general relativity the metric of spacetime is already fixed: there is no obvious sense in which one can make a decision which would change the metric to one's future from what it would otherwise have been.

To motivate the following, consider the case of a test field evolving on a fixed spacetime background, and let $S$ be a Cauchy surface in the fixed spacetime. Then if the initial data determining the field on $S$ differ only in $I^+(p) \cap S$, for some event $p$, we might reasonably regard this difference as due to a decision made at $p$. The question is, how do we alter this to cope with the case where the initial data determine the spacetime?

Consider the spacetime $(M, g_{\alpha\beta})$, defined as the maximal development of some initial data set $(\Sigma, h_{ab}, K_{ab}, J_a, \rho)$, where the matter fields satisfy some equation of the form (2.2) with the ray cones lying inside the light cones. Let $p$ be some point to the past of $i(\Sigma)$, and let $K = J^+(p) \cap \Sigma$. Now, consider an initial data set that agrees with this one on $\Sigma \setminus K$, but differs inside $K$. This will have a maximal development $(M', g'_{\alpha\beta})$. As in the vacuum case, we obtain from corollary 2.5 the further result

**Corollary 4.1.** *With all quantities as defined above, there is an isometry from $D(i(\Sigma \setminus K)) \subset M$ to $D(i'(\Sigma \setminus K)) \subset M'$.*

All the following results will hold if we model the decision to attempt to construct a warp drive by such an alteration in the initial data on some Cauchy surface to the future of the decision.

If we wish, we can consider only changes in the initial data which result in a (near) isometry from $I^-(i(\Sigma \setminus K))$ in $M$ to $I^-(i'(\Sigma \setminus K))$ in $M'$, so that the spacetimes initially look as nearly isometric as we wish, and then begin to diverge noticeably to the future of $p$. However, since the result to be established will hold for unrestricted changes to the initial data in $K$, it will *a forteriori* hold under any constraint made to those changes. We will therefore eschew any further discussion of such constraints.

Note that no requirement has been made on the relationship between $I^+(i(\Sigma))$ in $M$ and $I^+(i'(\Sigma))$ in $M'$; indeed, we should wish it to be possible for the mapping to be far from an isometry there. The kind of situation this should model is where some large-scale machinery has been constructed, and the decision of whether or not to set it in motion is made at some event $p$.

So now, let us consider the consequences of this for the attempt to construct a warp drive. First, consider the spacetime $M$. Let $A$ be a star (modelled as a point) whose world-line, $L$, intersects $i(\Sigma \setminus K)$. Then there are two possibilities: either $L$ remains inside $D^+(i(\Sigma \setminus K))$, or it eventually leaves this domain of dependence. In the former case, there is no causal path in $M$ from $p$ to $L$; in the latter, it is possible to travel from $p$ to $L$ and the earliest point on $L$ one can arrive at is the intersection of $L$ with the boundary of $D^+(i(\Sigma \setminus K))$.

Now, there is an isometry from $D(i(\Sigma \setminus K)$ to $D(i'(\Sigma \setminus K))$; it follows that also in $M'$ either the worldline of $A$ remains within the domain of dependence of $i'(\Sigma \setminus K)$, or that the interval inside this domain of dependence is isometric to that inside $D(i(\Sigma \setminus K))$. Thus in neither case is it possible to reach $L$ any sooner than it is in $M$, in the sense that one cannot reach an earlier point on the worldline of $L$. It may, however, be possible to arrange spacetime in such a way that a slower space-ship can arrive with less elapsed proper time; more interestingly, it may be possible to arrange spacetime in such a way that the return trip may be made short as measured by an observer who remains on the Earth. This possibility is considered in more detail in Krasnikov (1998).

We see, then, that with this notion of what it means to 'change the metric', and if all matter satisfies an evolution equation of the form (2.2) with the ray cones inside the light cones, one cannot construct a warp drive which has the effect of making it possible to travel 'faster than light'. However, there remains the case of what happens if the matter is not of this type: is it possible for physically reasonable matter to avoid the consequences developed above?

Still assuming that any matter fields are described by an evolution equation of the form (2.2), we see that in order for a warp drive to be effective—i.e. for it to be possible to change the initial data on $K$ in such a way that the portion of $L$ in $I^+(i'(\Sigma \setminus K))$ is shorter than that in $I^+(i(\Sigma \setminus K))$, the ray cone of the evolution equation (2.2) cannot lie inside the null cone of the metric $g_{\alpha\beta}$. But this means that the matter field can support the propagation of a signal at a speed exceeding that of light; however (Hawking and Ellis 1973), this is only possible if the matter violates the dominant energy condition. It therefore follows that a warp drive can only be constructed in the presence of exotic matter, confirming the suspicion voiced in Alcubierre (1994).

This result can be made rather more vigorous; as long as the initial data on $\Sigma \setminus K$ are unchanged, and the matter satisfies the evolution equation (2.2) with ray cones lying inside the light cones, $I^+(\Sigma \setminus K))$ is unchanged. Thus, whatever constraint one chooses on the data in $K$ to provide some interpretation of a free will decision, the result will hold.

In fact, we can go still further. In the case where $L$ eventually departs $I^+(i(\Sigma \setminus K))$ in $M$, there is a compact set $V$ containing $L \cap I^+(i(\Sigma \setminus K))$. Then as long as the initial data on $\Sigma \setminus K$ change sufficiently little, the metric on $V$ will also be changed by little, by the continuity theorem given in Hawking and Ellis (1973). So, even if we allow the initial data outside $K$ to change, by making the change sufficiently small, we can ensure that the proper time the star $A$ spends in $I^+(i'(\Sigma \setminus K))$ is very near that in the original spacetime $M$.

So, gathering this together, and recalling that those matter fields whose evolution is not determined by a symmetric hyperbolic system or which fail to satisfy the dominant energy condition are by definition (2.3) exotic, we have the result:

**Theorem 4.2.** *In the absence of exotic matter, it is impossible to construct a warp drive.*

## 5. Conclusions and discussion

We have seen that in the absence of exotic matter, the speed of light is in a well-defined sense a speed limit in general relativity. In order to do this it was necessary to make explicit the sense in which one might talk about the propagation of perturbations in spacetime, rather than that of test fields or test particles. This approach could be applied both in the case of vacuum spacetimes and those containing matter.

It is also possible to use the ideas exposed above to extend some other discussions beyond the realm of the behaviour of test particles.

For example, it is clear that a test particle inside the event horizon of maximally extended Schwarzschild space, $\mathcal{S}$, cannot escape. It is less obvious that a non-infinitesimal gravitational wave or physical particle cannot do so, since such objects affect the spacetime itself. However, if we consider a Cauchy surface $\Sigma$ for $\mathcal{S}$ which enters region II (in the notation of Hawking and Ellis 1973), we can consider an alternative spacetime defined by changing the initial data on $\Sigma$ on a subset, $K$, whose closure lies inside region II. From the above theorems, $D^+(\Sigma \setminus K)$ is unchanged by this alteration, so that some neighbourhood of the event horizon and the entirety of the external region to the future of $\Sigma$ are unchanged. This gives a simple way of seeing that unless the collapsing matter is exotic, once it has fallen inside the apparent horizon, the subsequent external solution is just external Schwarzschild. Similar comments will, of course, hold for the Kerr and Reissner–Nordström solutions.

This can even be interpreted as a kind of mini-censorship theorem, telling us that any singularity developing from a physically reasonable matter distribution inside the apparent horizon of an initial data surface for a black hole will lie inside the event horizon of the future development; i.e. it cannot be observed from the asymptotically flat region. (Recall that the classical singularity theorems only tell us that in the presence of a closed trapped surface, a singularity must develop—they do not tell us where the singularity will be.)

## Acknowledgment

I thank the anonymous referees for their suggestions for the improvement of the presentation of this paper.

## References


Alcubierre M 1994 The warp drive: hyper-fast travel within general relativity *Class. Quantum Grav.* **11** L73–7

Courant R and Hilbert D 1962 *Methods of Mathematical Physics* vol II (New York: Interscience)

Fischer A E 1995 Models of the Universe *Gen. Rel. Grav.* **28** 51–68

Fischer A E and Marsden J E 1979 The initial value problem and the dynamical formulation of general relativity *General Relativity, an Einstein Centenary Survey* ed S W Hawking and W Israel (Cambridge: Cambridge University Press)

Hawking S W and Ellis G F R 1973 *The Large-scale Structure of Spacetime* (Cambridge: Cambridge University Press)

Krasnikov S V 1998 Hyper-fast interstellar travel in general relativity *Phys. Rev.* D **57** 4760

Pfenning M J and Ford L H 1997 The unphysical nature of ‘warp drive’ *Class. Quantum Grav.* **14** 1743–51

Wald R M 1984 *General Relativity* (Chicago, IL: University of Chicago Press)