AAPT PHYSICS EDUCATION
AMERICAN JOURNAL of PHYSICS

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Citation: Am. J. Phys. **64**, 1517 (1996); doi: 10.1119/1.18415
View online: http://dx.doi.org/10.1119/1.18415
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v64/i12
Published by the American Association of Physics Teachers

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[16]Another conceivable linking transition that was put forward in discussion was a "median" transition from state 1 to 2 that fell midway between an isothermal and an adiabatic (i.e., $P_{2\text{median}}=\frac{1}{2}(P_{2\text{isotherm}}+P_{2\text{adiabat}})$ for given V_2. Another alternative that was sought was a combination of an "equality" transition, having ΔW_{out} equal to ΔQ_{in}, linked to an isothermal (for which the same property holds) that completes the loop. These possibilities seem not to have been introduced into the literature; a detailed examination of such "unorthodox" transitions may form the subject of a later paper.
[17]Of course, if the net work done is represented by the area of the main loop, it becomes 112 J which, taken in conjunction with the 224 J of heat intake in the individual cycles, constitutes a 50% efficiency for this engine.

A pedagogical model of primordial helium synthesis

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(Received 28 June 1995; accepted 11 March 1996)

The calculation of the primordial hydrogen and helium abundances in the big-bang cosmology is presented in an oversimplified model accessible to university physics students who have had no physics beyond an elementary modern physics course. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

At Southern Nazarene University we frequently offer a one-credit-hour course in big-bang cosmology. The only prerequisite is prior completion of our two-semester calculus-based introductory physics course, which includes an introduction to elementary 20th century physics. Our cosmology class therefore includes many sophomores, along with juniors and seniors. Can we give the sophomores a simplified quantitative model of big-bang primordial nucleosynthesis, illustrating the basic concepts while glossing over the messy details? If so, then in their future study our students are in a position to better appreciate more realistic models.

Extending the students' knowledge from introductory physics to the kinematics and thermodynamics of the early universe is fairly straightforward, if one takes as given the equations of motion for the cosmic scale factor $R(t)$ (see the Appendix). More challenging is finding arguments that connect the nucleosynthesis calculations to these students' present conceptual system. In the very early universe there are many nuclear species transforming into one another, described by a set of coupled rate equations. The calculation of the abundances of the neutrons and protons from which they are made involves a rate equation with time-dependent coefficients. These rate coefficients, determined from quantum statistical mechanics and the details of the weak interaction, are temperature dependent, and thus time dependent because the universe cools as it expands. Clearly, a full treatment of this problem is beyond the scope of the average college sophomore.

Within the literature on this subject[1–10] there are already at least two models advertised as "simplified."[11,12] In addition, there exist many fine textbooks[13–16] from which one can extract the flow of ideas. But these resources are still beyond the reach of students who have just completed the calculus-based introductory physics course, students who have not yet had a course in differential equations, and have received no instruction in quantum statistical mechanics or the art of calculating weak interaction cross sections. What is "simple" is a function of one's experience. We construct here an oversimplified model that engages the introductory physicist in a quantitative discussion of big-bang nucleosynthesis.

Let us profile the background assumed of the student. The sophomore student we have in mind recently completed the calculus-based introductory university physics course, and was either introduced to elementary 20th Century physics there[17] or has recently taken an elementary modern physics course.[18] Of relativity, our student knows that a free particle's total energy is mc^2 plus kinetic energy, and that the "spacetime interval" is invariant. Of nuclear physics, our student is familiar with the exponential behavior of radioactive decay, knows of beta decay and its inverse reactions, and has met the proton–proton cycle of helium fusion. Of statistical mechanics, our student appreciates that for a many-particle system in thermal equilibrium at temperature T, the average energy per particle is of order kT, and that the probability of a microscopic system having energy E in this environment is proportional to $e^{-E/kT}$. Of cosmology, perhaps from general reading such as Steven Weinberg's *The First Three Minutes*,[19] our student has learned that the universe is expanding according to Hubble's law. This is all the background we require of the student to whom we present the model of Sec. II.

In Sec. II we conceptualize our oversimplified model and put it into elementary mathematics. We find that everything can be connected to the student's system of concepts described above except two temperatures: the temperature where the weak interaction rate can no longer keep up with the Hubble expansion rate, and the temperature where hydrogen fuses to helium. Of course, it is in the determination of these temperatures where most of the work is done in the realistic models, and our model neatly sidesteps this genuinely hard work. But this is precisely what a "pedagogical" model is supposed to do: make the student familiar with the *concepts* (which are usually simple), and prepare the mind to appreciate the *technical details* (which are usually complicated). We thereby hope to inspire a student to take the next step and try one's hand at the realistic model—and perhaps, one day, even improve upon it. As has been observed in another context, "The same knowledge that makes the goal conceivable also drives home how hard it is to achieve."[20]

But there are still those two temperature "gaps" in our oversimplified model. However, they are gaps only for our sophomore student profiled above. For the upper-division physics student who is prepared to go farther, those gaps can be filled in, at least roughly using junior- or senior-level statistical mechanics. This is done in Sec. III. In Sec. IV we indicate, as is told to a student, how one extends the model to include other isotopes beyond hydrogen-1 and helium-4. Section V is a visual comparison of this model to the realistic ones. In the Appendix appear the elementary cosmological arguments that were previously developed in our sophomore cosmology syllabus, developments we also need for the nucleosynthesis calculation.

II. THE MODEL

The only baryons considered are protons and neutrons. We model the baryonic composition of the early universe as a two-state system (see Fig. 1), where the neutron state is the excited state because the neutron is slightly more massive than the proton. Neutrons and protons can change into one another through the weak interactions between the baryons and leptons,

$$p+e^- \leftrightarrow n+\nu \tag{1}$$

and

$$p+\bar{\nu} \leftrightarrow n+e^+. \tag{2}$$

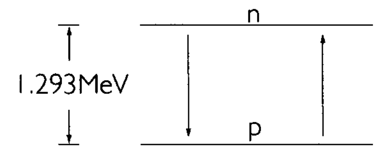

Fig. 1. The baryon is modeled as a two-state system. The neutron state is the excited state because the neutron mass is 1.293 MeV/c^2 greater than the proton mass. The transitions between the states proceed according to Eqs. (1)–(3).

In addition, because of its larger mass a free neutron may also spontaneously decay into a proton,

$$n \rightarrow p+e^- + \bar{\nu}. \tag{3}$$

We neglect the inverse reaction of Eq. (3), assuming that the occurrence of three-body collisions is extremely rare in comparison to two-body collisions. We also ignore the other leptons beyond the electron, its neutrino, and their antiparticles.

Let us assume that the number of baryons, denoted by N, is conserved. This means we are jumping into the story of the very early universe after the reactions that do not conserve baryon number have tipped the composition of the universe in favor of matter over antimatter (viz., after about 0.01s following the big bang). If N_n is the number of protons and N_p is the number of neutrons, then

$$N_n+N_p=N. \tag{4}$$

This can be written in terms of the fractions $X_i \equiv N_i/N$ of the baryons that are species i,

$$X_n+X_p=1. \tag{5}$$

Our goal is to calculate mass fractions of baryonic matter that emerge from the big bang as hydrogen-1 and as helium-4. *We ignore all other isotopes*. This is justified in the first approximation because hydrogen-1 and helium-4 account for most of the nuclei in nature. Their abundances are determined by the value of X_n at the time of nucleosynthesis. Let us illustrate why this is so.

Suppose that when nucleosynthesis occurs the neutron fraction X_n turns out to be 1/8. Then the proton fraction X_p is 7/8, and the neutron-to-proton ratio X_n/X_p is 1/7. This means that for every pair of neutrons there are 14 protons. Of these 16 nucleons, the two neutrons combine with two protons to make a nucleus of helium-4, leaving the remaining 12 protons as nuclei of hydrogen-1. In this illustration the primordial baryonic mass of the universe is 4/16=25% as He^4, and 12/16=75% as H^1.

Generalizing this example, the mass fraction Y of baryons that forms into helium-4 is

$$Y=(\#He^4 \text{ nuclei})(4m)/[N_pm+N_nm]$$
$$=4(\#He^4 \text{ nuclei})/N, \tag{6}$$

where Eq. (4) has been used along with the simplification that the proton and neutron have roughly the same mass m, and the mass of helium-4 is approximately $4m$. Because the number of helium nuclei is one-half the number of neutrons, Eq. (6) becomes

$$Y=2X_n. \tag{7}$$

So, if we know X_n at the time of nucleosynthesis, then we know the mass fraction Y that is helium-4, and the mass fraction $1-Y$ that is hydrogen-1. Our task, then, is to calculate the value of X_n at the time of nucleosynthesis. We proceed as follows.

(1) The universe at the times of interest is initially a gas of photons, electrons and positrons, neutrinos and antineutrinos, and neutrons and protons. We ignore all other particles in our simplified model. This gas is exceedingly hot and expanding, and cools as it expands. We model this gas of particles as a system in thermal equilibrium at temperature T.

(2) When a system is in thermal equilibrium, the probability that a particle is in a state of energy E is proportional to $e^{-E/kT}$. The energy of a gas particle is $K+mc^2$, where K is the kinetic energy. Since the kinetic energy is determined by the temperature, the average kinetic energies of the protons and neutrons are the same. Because the fractions X_n and X_p are the probabilities that a baryon is in the neutron or proton state respectively, and noting that the mass of the neutron is 1.293 MeV/c^2 (~1.3 MeV/c^2) greater than the mass of the proton, it follows that

$$X_n/X_p=\exp(-1.3\text{MeV}/kT). \tag{8}$$

Equations (5) and (8) can be combined into

$$X_n=1/[1+\exp(1.3\text{MeV}/kT)] \tag{9}$$

from which we see that X_n is determined by T. We must turn to the thermodynamics, elementary particle physics, and nuclear physics in the early universe to determine T at times of interest. Note at very early times, when the temperature is so high that $1/kT$ is indistinguishable from zero, that $X_n \approx X_p \approx 1/2$.

(3) We will need to relate temperature to time t. The equations describing the expansion of the universe were developed early in our course in an argument designed for sophomores, and are summarized in the Appendix. There it is shown that in our simple model (using numbers faithful to this model) the relation is

$$T^2 t=1.9\times 10^{20}\ \text{K}^2\ \text{s}. \tag{10}$$

(4) Because the temperature is decreasing, we see from Eqs. (9) and (5) that X_n is decreasing and X_p is increasing. It is instructive to write a differential equation for the evolution of X_n even though solving it is beyond the scope of our sophomore-level course. This differential equation, and the more realistic extensions of it which include all the nuclear species involved, is the heart of the matter, and writing it here gives the student a glimpse of where this subject begins to get complicated.
The rate at which X_n increases is proportional to X_p because neutrons are made from protons; and X_n decreases at a rate proportional to X_n because neutrons must already exist before they can be converted into protons. Hence we may write

$$\frac{dX_n}{dt}=\lambda(p\rightarrow n)X_p-\lambda(n\rightarrow p)X_n, \tag{11}$$

where $\lambda(p\rightarrow n)$ and $\lambda(n\rightarrow p)$ are rate coefficients. These rate coefficients depend on the energies of the particles, and thus upon the temperature of the gas in which they find themselves. Because the temperature falls as the universe expands, these coefficients are time dependent. They are determined as functions of T by applying the details of weak interaction physics and statistical mechanics to our system.

(5) The $p\leftrightarrow n$ reactions of Eqs. (1) and (2) are driven by leptons colliding with the baryons. These reactions will cease whenever the universe has expanded and cooled such that the particles are carried apart by the Hubble expansion faster than they react by collision. If we write the reaction rate in terms of temperature, and write the Hubble parameter (which measures the expansion rate) in terms of temperature, then set the rates equal to one another, we will have a good estimate of the temperature of this quenching. Then Eq. (10) will tell us the time when this "freeze-out" occurs. In our simple model let us suppose this freeze-out occurs instantaneously when the corresponding temperature is reached.
We must for now take it as given that this freeze-out occurs at $T\approx 1\times 10^{10}$ K, or $kT\approx 0.86$ MeV. From Eq. (10), this temperature corresponds to the freeze-out time $t_f=1.9$ s. At this instant, Eq. (9) tells us that the neutron abundance is

$$X_n(t_f)=1/[1+\exp(1.3\text{MeV}/0.86\text{MeV})]\approx 0.18. \tag{12}$$

(6) In our model, after the time t_f the only other baryonic transformation that occurs is the radioactive decay of neutrons into protons. From the Particle Data Group tables[21] we look up the lifetime of the free neutron, and find it to be 887.0±2.0 s. Thus for $t\geqslant t_f$,

$$X_n(t)=X_n(t_f)\exp[-(t-t_f)/887\text{s}]$$
$$=0.18\ \exp[-(t-1.9s)/887\text{s}] \tag{13}$$

is the fraction of baryons that are neutrons after freeze-out, but before the neutrons and protons combine to form helium-4 nuclei.
At freeze-out, the electrons and positrons annihilate to produce more photons, warming the gas of photons and charged particles, while not affecting the neutrinos. But that is another story which represents a departure from a simple T vs t relationship such as Eq. (10). We do not pursue this topic in our oversimplified model, but merely point it out as another issue which a more realistic model must address.

(7) At this point in the model we have a gas consisting of radioactive neutrons and stable protons immersed in a gas of photons and neutrinos. Let us recall how helium-4 nuclei are built out of protons and neutrons. One of the helium fusion reactions is the following (where d is the deuteron and γ is the photon):

$$\rho+p\rightarrow d+e^{+}+\nu, \tag{14}$$

$$d+p\rightarrow \text{He}^3+\gamma. \tag{15}$$

These reactions must happen twice, so that the final reaction may happen once:

$$\text{He}^3+\text{He}^3\rightarrow \text{He}^4+2p \tag{16}$$

giving for the overall reaction $4p\rightarrow \text{He}^4+2e^{+}+2\nu+2\gamma+25$ MeV. Since these nuclei are in a very hot gas, collisions (generally with photons) will fission these nuclei as quickly as they are formed, until the energy of the photon gas drops to some sufficiently small value. (The temperature where that happens depends on the ratio of the number of baryons to photons; hence in a realistic model the answer depends on this ratio.) Just as a chain is no stronger than its weakest link, likewise helium-4 cannot form until the temperature has dropped below the photo-fission threshold of the most loosely

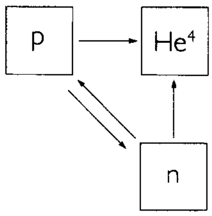

Fig. 2. These are the reactions considered in our oversimplified model. Contrast this scenario to that of Fig. 3.

bound intermediate nucleus, the deuteron. We cannot find this temperature merely by setting kT equal to the binding energy of the deuteron (~2.2 MeV)—there is more to it than that. Here, then, is the second place where the number we need is beyond the scope of a sophomore physics class, although the number can be derived by a more advanced undergraduate who has studied statistical mechanics (see Sec. III). To cut a long story short, the sequence of fusion reactions will be immune to photo-fission when the temperature drops to about $T=0.8\times10^9$ K, which according to Eq. (10) occurs at the time $t=297$ s. Then the surviving neutrons will bind with protons to form nuclei. To keep our model simple, we suppose that when this temperature is reached, the fusion of protons and neutrons to helium-4 occurs instantaneously. (This is an approximation, since during the nucleosynthesis process there will be various nuclear species and free protons and neutrons coexisting, and with multiple reaction paths.) Since the fraction of neutrons has been monotonically decreasing and the fraction of protons has been increasing (both fractions starting from 1/2), *all* the neutrons are used to form helium-4 nuclei, and all unattached protons are by default the nuclei of hydrogen-1, these being the only isotopes considered in our model. Thus from Eq. (13), the fraction of neutrons that exist at the time of nucleosynthesis and thereafter is

$$X_n=0.18\ \exp[-(297\text{s}-1.9\text{s})/887\text{s}]\approx0.13. \quad (17)$$

From now on, no more reactions occur that convert neutrons to protons, since within a stable nucleus the numbers of protons and neutrons remains constant.

Using the value of X_n given by Eq. (17), Eq. (7) now tells us the fraction of baryonic mass that emerges from the big bang as helium-4:

$$Y=0.26, \quad (18)$$

leaving 0.74 as the fraction of baryonic mass that is hydrogen-1. This result is to be compared to observations and to realistic models. Olive and Steigman[22] analyze observations of the helium-4, nitrogen, and oxygen abundances for some four dozen extragalactic, low-metallicity H_{II} regions (viz., extragalactic clouds of hot, ionized hydrogen gas), then extrapolate the He^4/N, N/O, and He^4/O ratios to zero metallicity to infer the primordial helium-4 mass fraction. They obtain from the comparisons to nitrogen and oxygen data the value

$$Y_{\text{obs}}=0.232\pm0.003(\text{stat})\pm0.005(\text{syst}). \quad (19)$$

Including all statistical and systematic uncertainty, their analysis places the maximum upper bound for Y at 0.243. Similarly, the realistic models calculate results[23] such as $0.235\leq Y\leq0.245$, and $0.236\leq Y\leq0.243$. Taking $Y\approx0.24$ as a generous "observed value" for comparison to our oversimplified model for which $Y=0.26$, we find the latter to be about 8% larger than the former.

We should not be astonished that our model does not agree out to two significant figures with the serious assessments, since we have made a deliberately oversimplified model whose purpose is to merely *illustrate the calculation in general-physics terms*. Encouragingly, our pedagogical model errs on the side we would expect: having neglected the production of all isotopes other than hydrogen-1 and helium-4, we should expect our Y to be somewhat too large, and play the role of an upper limit, since the neutrons that might have gone into such nuclei as deuterons, tritium, helium-3 or lithium are, in our simple model, found in helium-4 instead. However, the result of our oversimplified model comes close enough to the literature values of Y so that when our students begin to study the standard texts and papers that describe realistic models, they will feel somewhat at home in that literature.

The simplicity of the helium-4 and hydrogen-1 abundance calculation (given the two temperatures) shows that the mettle of realistic calculations is tested in their predictions of the abundances of the other isotopes of hydrogen and helium, plus beryllium, lithium, and the other light primordial elements.

III. THE TWO TEMPERATURES

In our oversimplified model, all steps except two are connected to our students' conceptual world. The two points of input that cannot be easily justified with sophomore-level arguments are the two temperatures quoted above: (1) the weak interactions freeze out at $T\approx1.0\times10^{10}$ K, and (2) nucleosynthesis occurs at $T\approx0.8\times10^9$ K. Here we show how these two temperatures may be roughly justified through upper-division undergraduate statistical mechanics.

We wish to calculate the rate of the reactions of Eqs. (1) and (2).[24] This rate is the product of the number density of leptons n_l with which the baryons must collide, and the weak interaction cross-section σ_{wk} (and also c in conventional units). The student who has taken upper-division statistical mechanics knows that for these fermions of mass m, the number density $n_l(p)$ of particles with momentum of magnitude in the interval from p to $p+dp$ is (with $\hbar=c=1$)[25]

$$n_l(p)dp=\pi^{-2}p^2dp\{1+\exp[(p^2+m^2)^{1/2}/kT]\}^{-1}. \quad (20)$$

In the very early universe, all particles are highly relativistic, so that all masses are negligible. Integrating over all momentum, in the relativistic regime the total number density becomes $n_l\approx(kT)^3$.

Next we turn to the cross-section. The calculation of weak-interaction cross-sections is not usually attempted in undergraduate course work. However, there are references that are accessible to the motivated undergraduate.[26] The result is simple: σ_{wk} is proportional to $G_F^2q^2$, where q^2 is the four-momentum squared of the incoming lepton and G_F is the Fermi coupling constant, $G_F=10^{-5}\ \text{GeV}^{-2}$ (with $\hbar=c=1$). In terms of temperature, the cross section is approximated by[24] $\sigma_{\text{wk}}\approx G_F^2(kT)^2$, so that the reaction rate is $n_l\sigma_{\text{wk}}\approx G_F^2(kT)^5$.

The other half of our freeze-out calculation is to express the Hubble parameter $H\equiv(dR/dt)/R$ (where R is the cosmic scale factor) in terms of temperature. This portion of the

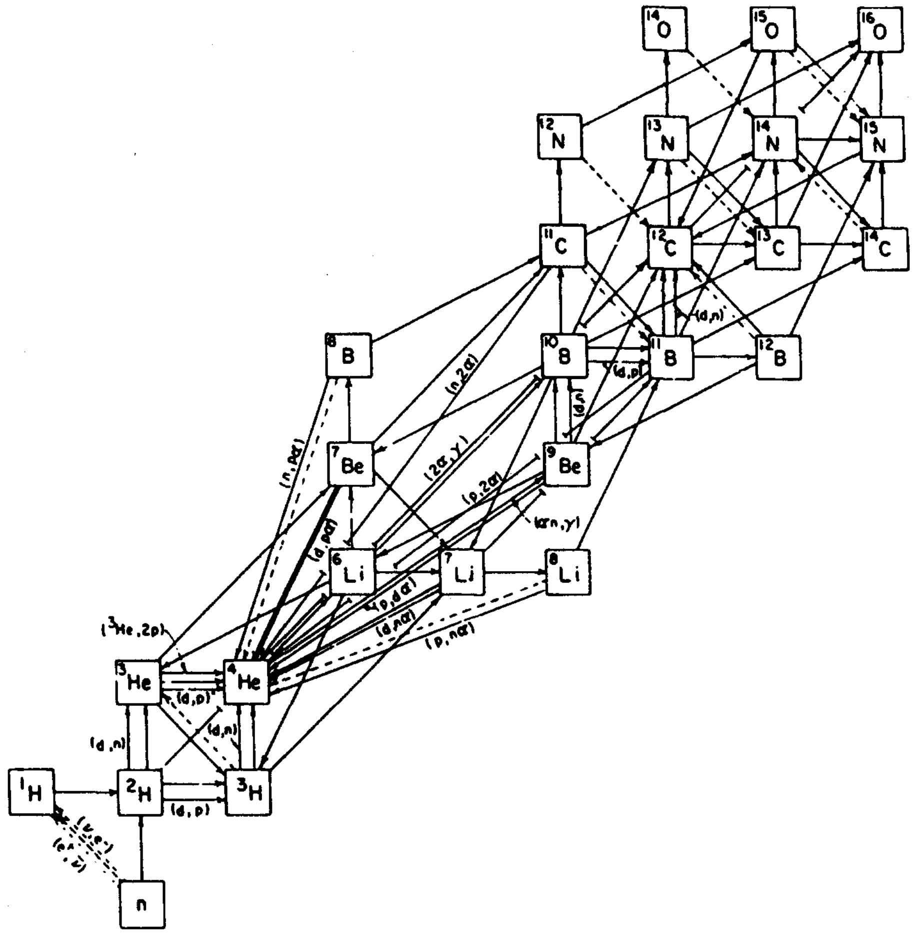

Fig. 3. These are the reactions considered in realistic models. For instance, the line going from He^4 to C^{12} denotes the triple-α fusion reaction $He^4+He^4+He^4 \rightarrow C^{12}+\gamma$. From D. Schramm and R. V. Wagoner, "Element Production in the Early Universe," Annu. Rev. Nuc. Sci. **27**, 37-74 (1977). Adapted, with permission, from the *Annual Reviews of Nuclear Science*, Volume 27, © 1977, by Annual Reviews, Inc.

calculation can be completed with our class of sophomores profiled in Sec. I. From Eq. (A4) in the Appendix, neglecting the curvature parameter k', and using an energy density given by Stefan's law, we find that $H \approx G^{1/2}(kT)^2$ ($G \equiv$Newton's gravitational constant $\approx 10^{-38}$ GeV^2). When the reaction rate is equated to the expansion rate, we find that $kT \approx 10^{-3}$ GeV, or $T \approx 1 \times 10^{10}$ K.

(2) To obtain the temperature where neutrons and protons fuse into helium-4, we proceed as follows.[27] Again, the results depend on upper-level undergraduate statistical mechanics. The number density n_i of nuclei of charge Z_i, mass m_i, and number of baryons A_i is in the non-relativistic Maxwell–Boltzmann approximation (with $\hbar = c = 1$),

$$n_i = g_i(m_i kT/2\pi)^{3/2} \exp[(\mu_i - m_i)/kT], \tag{21}$$

where g_i counts the spin degrees of freedom. The chemical potentials μ_i can be eliminated in favor of those of the proton and neutron [using expressions similar to Eq. (21) for free protons and neutrons], to derive an expression for the mass fraction contributed by nuclear species i to the baryonic composition of the universe. The mass fraction X_i at the time the nucleosynthesis is $n_i A_i/n_N$, where n_N is the number density of all the baryons. After some algebra, one obtains

$$X_i = 1/2 X_p^{Z_i} X_n^{A_i} g_i A_i^{1/2} [\zeta(T)]^{A_i - 1} \exp[B_i/kT], \tag{22}$$

where B_i is the binding energy of nucleus i, and $\zeta(T) \equiv 4\pi^3 n_N (2\pi m_N kT)^{-3/2}$, which is very small in the early universe because T is large. When T has dropped to a value sufficiently low that the temperature-dependent factors of Eq. (22) are of order unity (justifying the Maxwell–Boltzmann statistics), viz., when

$$T \approx B_i/[k(A_i - 1)|\ln \zeta(T)|], \tag{23}$$

then the mass fraction X_i becomes appreciable. Solving this equation numerically estimates the temperature at which nuclide i begins to form.[28]

IV. POINTING THE STUDENT TOWARDS THE MORE REALISTIC MODELS

The more realistic models must take into account (a) the additional generations of leptons, and (b) the fact that in the very early universe there were *many* nuclear species coexisting, requiring a large set of coupled rate equations in the same spirit as Eq. (11). Let us briefly review these features here, to show our student where the realistic nucleosynthesis calculations become complicated.

(a) The other leptons include the muon and tau, plus their neutrinos. Consider the muon, whose mass is about 207 times the electron mass. When the temperature drops below the threshold temperature of the muon defined by $T = m_\mu c^2/k \approx 10^{12}$ K, then the ratio of the weak interaction reaction rate to the expansion rate is damped by the factor $\exp(-m_\mu c^2/kT)$, so that $n_l \sigma_{wk}/H \approx (T/10^{10}\text{ K})^3 \exp(-10^{12}$

K/T). The muons (and thus their neutrinos) drop out of the reactions when this quantity becomes of order unity, or $T \approx 1.3 \times 10^{11}$ K.[29] When a species of charged lepton drops out of reaction, its annihilation with its antiparticles re-heats the gas of photons and surviving charged particles, introducing departures from Eq. (10) (or its more numerically accurate counterpart in the realistic models). This feature is taken into account through the thermodynamics of the very early universe. For instance, when the electrons and positrons drop out of reaction, the conservation of entropy predicts that the temperature of the photon and charged particle gas is greater than the intermingled gas of neutrinos by a factor of $(11/4)^{1/3} \approx 1.4$. The 11/4 is traced back to the spin degrees of freedom of the various particles.[30]

(b) We mentioned earlier how each transition between species (from the proton and neutron through all isotopes of the light elements) represents a term in a differential rate equation. For nuclear species i (where i denotes the list protons, neutrons, and the various isotopes of hydrogen, helium, lithium,...) which interacts with other species with nuclear reaction rates $\Lambda(i \rightarrow j)$, the rate equation for the number fraction X_i is

$$\frac{dX_i}{dt} = -X_i \sum_j \Lambda(i \rightarrow j) + \sum_j \Lambda(j \rightarrow i) X_j, \tag{24}$$

where

$$\sum_j X_j = 1. \tag{25}$$

The rate coefficients $\Lambda(i \rightarrow j)$ come from nuclear physics measurements. Finally, the set of coupled Eqs. (24) must be solved numerically. This is a formidable *technical* task. Perhaps our oversimplified model will make it visualizable *conceptually*.

In our simple model, the nucleosynthesis of two neutrons and two protons into helium-4 is assumed to be instantaneous. In reality, it is not. Helium-4 (and other nuclei) will co-exist with free protons and neutrons for some time. This is built into the numerical integrations of the coupled reaction rate equations.

The realistic models predict the mass fraction of deuterium, tritium, and helium-3 are on the order of 1×10^{-12}, 2×10^{-19}, and 5×10^{-19}, respectively. Lithium-7, though produced in only trace amounts, is a sensitive test of models of primordial synthesis. Of course, our pedagogical model omits these nuclei entirely, and the real challenge of nucleosynthesis is predicting the abundances of these species.

V. SUMMARY

In the tradition of an introductory university physics course or text, we have attempted to construct a quantitative argument that provides insight into an important physical process while using only elementary concepts, where by "elementary" we mean here "at the level of a sophomore modern physics course."

Let us visually compare our oversimplified model to the more realistic ones. In Fig. 2 we sketch a "flow chart" of the reactions considered here. It shows protons and neutrons reacting with one another, then all of the neutrons joining with some of the protons to produce helium-4. This is a considerable simplification of the flow chart of reactions used in the more complete model, which the reader can find in the literature,[31] which is reproduced in Fig. 3.

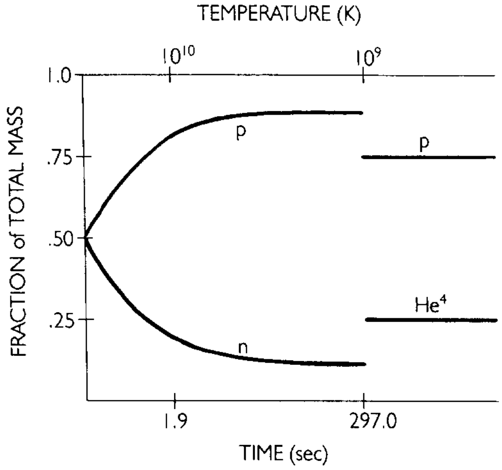

Fig. 4. Fraction of total mass for protons, neutrons, and helium-4 as functions of time, according to our oversimplified model. The origins of both axes are at their intersection. X_n and X_p as introduced in the text are the *number* fractions of neutrons and protons respectively, but before helium synthesis they are approximately the *mass* fractions as well since the neutron and proton mass are the same to roughly one part in 1000. Contrast this scenario to that of Fig. 5.

In Fig. 4 we sketch the evolution of the mass fractions of all species considered in our model. (Since the neutron and proton mass differ by less than two parts out of a thousand, X_n and X_p are approximately the mass fractions of free neutrons and protons in the model.) The simplicity of this figure is to be contrasted to the corresponding diagram in the realistic model,[32] reproduced in Fig. 5.

It is worthwhile to point out to the student that the realistic primordial nucleosynthesis calculations relate Y to three other parameters in the model: the number of lepton generations N_ν, the neutron lifetime τ_n, and the baryon-to-photon ratio η ($\eta_{10} = 10^{10}\eta$). Walker *et al.*,[33] for example, display

$$Y = 0.228 + 0.010 \ln \eta_{10} + 0.012(N_\nu - 3) + 0.185(\tau_n/889\text{s} - 1). \tag{26}$$

The hot big-bang nucleosynthesis problem is an important and interesting one for students to be aware of early in their careers. There is currently a debate as to whether there is a discrepancy, on the order of a few parts in a thousand, between the value of the primordial helium abundance and the value calculated from primordial nucleosynthesis models[34,35] that cannot be resolved without pushing one of the parameters N_ν, τ_n, and η slightly beyond their experimental values. The observed value is determined from measurements of extragalactic H_{II} regions, numbers which must then be extrapolated through stellar fusion models to obtain primordial helium abundance estimates. None of this is easy, and there are many uncertainties. If beginning physicists—students in their sophomore year of study—are introduced to an oversimplified model that provides them with a manageable but authenic introduction to an intriguing and important question that presently exists at the research frontier, they may be inspired to learn as much as possible, as quickly as they can, to try their hand at helping to answer the problem when they have learned more. It is encouraging to see how far one can get into the problem using only sophomore arguments. The overall flow of ideas is quite simple. Seeing this simplicity is the important first impression that should be made in any modeling exercise.

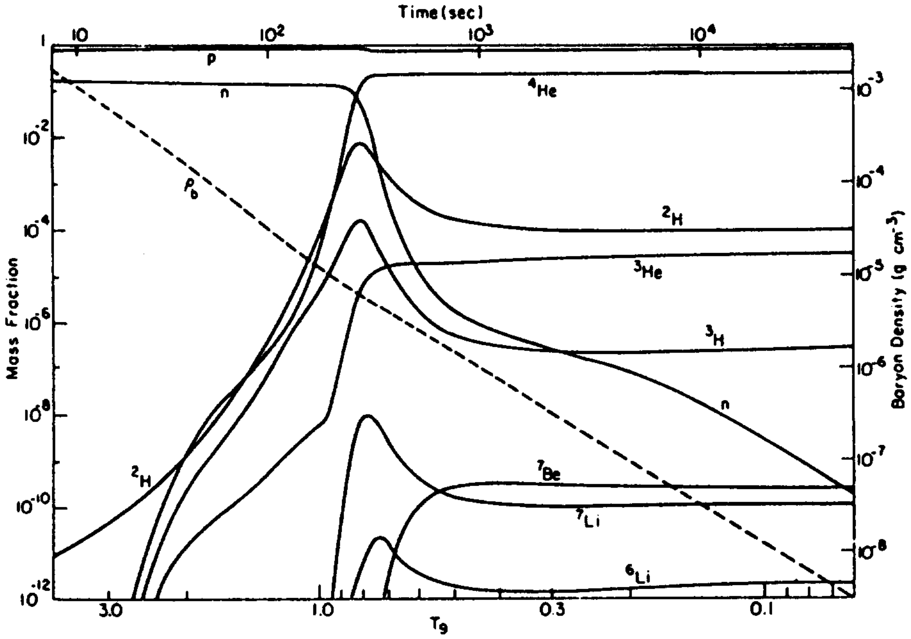

Fig. 5. Fraction of total mass of for protons, neutrons, and light nuclei, as function of time according to realistic models. From D. Schramm and R. V. Wagoner, "Element Production in the Early Universe," Annu. Rev. Nuc. Sci. **27**, 37–74 (1977). Adapted, with permission, from the *Annual Reviews of Nuclear Science*, Volume 27, © 1977, by Annual Reviews, Inc.

This paper began as a student project for BE, with DN as the faculty sponsor. The student co-author was a junior physics and mathematics double major when the project began. To the person approaching the primordial nucleosynthesis calculation for the first time, the welter of detail, even when summarized in excellent texts such as Refs. 13–16, is bewildering. Like a rushed introductory physics course, the complexity of events and the many numbers being thrown about is to the newcomer like drinking from the proverbial fire hose. In addition, we were preparing to offer our cosmology seminar again in the Fall '94 semester, in which there would be several sophomore students who had just completed the general physics course. In the past, this course described quantitatively the expansion of the universe and the background radiation, but merely noted the bottom-line results for the nucleosynthesis calculations. For the Fall '94 semester we wanted to include a quantitative discussion of the nucleosynthesis problem, and were thus motivated for this reason as well as BE's project to construct some oversimplified version of it. This paper is the result of that process that was worked out between BE, DN, and the cosmology class. It is offered here in the hope that other professors and students may wish to improve upon the sophomore-level discussion of the primordial nucleosynthesis problem.

ACKNOWLEDGMENTS

For their input and feedback, we thank the students of Southern Nazarene University who participated in the course Physics 1391, "Selected Topics in Physics: Cosmology." The course included a quantitative discussion of nucleosynthesis for the first time in the fall 1994 semester. Several of the students in the course fit the profile of Sec. I, having just completed the introductory university physics course the preceding semester. We also thank Professor Harvey Leff for his encouraging suggestions, two anonymous referees for much-appreciated criticisms and advice, and Annual Reviews Inc. for permission to reproduce figures.

APPENDIX: TEMPERATURE AS A FUNCTION OF TIME

We sketch a simple argument for the time-dependence of the temperature in the early universe. This is a summary of presentations made earlier in our sophomore-level cosmology course, when the expansion of the universe is connected to concepts familiar from introductory modern physics. From their introduction to special relativity in the general physics course, our students know about the invariance of the spacetime interval,

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) \tag{A1}$$

(with $c = 1$) or, stepping beyond the elementary modern physics course by switching to spherical coordinates,

$$ds^2 = dt^2 - (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2). \tag{A2}$$

Since the universe is expanding, the spatial coordinate grid is continuously rescaled by some factor $R(t)$; furthermore, since space is not necessarily Euclidian, we modify the interval to take into account the possibilities offered by non-Euclidian geometry, and write it as

$$ds^2 = dt^2 - R^2[(1 - k'r^2)^{-1} \, dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2], \tag{A3}$$

where k' is the curvature parameter: $k' = 0, \pm 1$. With this form of the interval, Einstein's field equations of general relativity tell us how the scale factor $R(t)$ evolves in time, although in our sophomore-level course we must accept them as given. One of these equations is

$$\left(\frac{dR}{dt}\right)^2 - \left(\frac{8\pi G}{3}\right)\rho R^2 + k' = 0, \tag{A4}$$

whose first two terms are analogous to kinetic and potential energy (where ρ is the energy density of the universe and G is Newton's gravitational constant). The other equation is

$$\frac{d^2R}{dt^2} = -\left(\frac{4\pi G}{3}\right)(\rho - 3P)R \tag{A5}$$

(where P is the pressure), which is analogous to Newton's second law, because it says that the expansion decelerates because of gravity. The analogies to Newtonian mechanics are pointed out because in our sophomore class we cannot derive from Einstein's field equations the connection between Eqs. (A4) and (A5) to the metric of Eq. (A3), so we can only connect them by analogy to something familiar.

In the very early universe, all particles are ultra-relativistic. Except for the number of spin degrees of freedom all particles are thus "photon-like," their masses being negligible. Hence the energy density is given by Stefan's law, $\rho = g\sigma T^4$, where g counts the number of spin degrees of freedom and σ is the Stefan constant (with polarization degrees of freedom divided out, $\sigma = 2.835\times 10^{-8}\ \mathrm{W/m^2\,K^4}$). In addition, it is an elementary exercise to show that in the early universe the temperature drops as $T \sim 1/R$, so we may write $R = \Lambda/T$ for some constant Λ. It is also straightforward to show that in the very early universe the neglect of k' is a valid approximation even if $k' \neq 0$ exactly.[36] Putting these points together, Eq. (A4) reduces to

$$\frac{R dR}{dt} = \mu\Lambda^2 \tag{A6}$$

where (in SI units) $\mu = [8\pi G g\sigma/3c^3]^{1/2}$. Integrating Eq. (A6), we obtain

$$R = (2\mu t)^{1/2}\Lambda. \tag{A7}$$

Since $T = \Lambda/R$, we find that

$$T^2 \approx 1.89\times 10^{20}\ \mathrm{K^2\ s}/t, \tag{A8}$$

where g has been taken to be 12 (two spin degrees of freedom each for the photon, electron, positron, proton, and neutron, one each for the neutrino and antineutrino, and approximating as unity the 7/8 that arises in the quantum statistics of fermions, since the explanation of the 7/8 is beyond the scope of our course).

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[13]S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), Chap. 15.

[14]P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton U. P., Princeton, NJ, 1993), Chap. 6.

[15]E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Reading, MA, 1990), Chap. 1.

[16]J. V. Narlikar, *Introduction to Cosmology* (Cambridge U. P., Cambridge, U.K., 1993), Chap. 5.

[17]Texts and curricula whose level is the same as that assumed of the student described in Sec. II of this paper include: (a) any curriculum of the Introductory University Physics Project, which field-tested experimental two-semester, calculus-based physics courses that were to include some meaningful 20th Century physics (the students in the SNU cosmology course had completed the IUPP model *Structures and Interactions*). The IUPP and its four models are described by J. S. Rigden, D. F. Holcomb, and R. DiStefano, "The Introductory University Physics Project," Phys. Today **46**, 32–37 (April 1993). (b) Any standard two-semester calculus-based general physics course using texts of the *Halliday & Resnick* genre, followed by a one-semester "modern physics" course using texts such as those of Refs. 18.

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[23]The two values quoted are from Refs. 9 and 10, respectively.

[24]For instance, see Ref. 13, pp. 533–535.

[25]The success of the introductory relativity text of E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Freeman, San Francisco, 1966), demonstrates that units such as $c=1$ are readily adapted by sophomores.

[26]See, e.g., I. J. R. Aitchison and A. J. G. Hey, *Gauge Theories of Particle Physics- A Practical Introduction* (Adam Hilger Ltd., Bristol, UK, 1982).

[27]For instance, see Ref. 13, pp. 551–552.

[28]A more detailed analysis depends on the photon-to-baryon ratio. When the number of photons in the tail of the Planck distribution having energies above the deuteron's binding energy drops below N, then nucleosynthesis begins. A fine elementary discussion of this point is found in D. Halliday, R. Resnick, and K. S. Krane, *Physics*, Volume 2 (Wiley, New York, 1992), p. 1207.

[29]See, e.g., Ref. 13, pp. 538–539.

[30]For instance, see Ref. 13, pp. 536–537.

[31]See Schramm and Wagoner, Ref. 7, p. 48.

[32]See, e.g., Schramm and Wagoner, Ref. 7 (1973), p. 51.

[33]Reference 10, p. 56. These authors quote 889 s as the neutron lifetime, using data older than that we use here.

[34]N. Hata, R. J. Sherrer, G. Steigman, D. Thomas, T. P. Walker, S. Bludman, and P. Langacker, "Big Bang Nucleosynthisis in Crisis," Phys. Rev. Lett. **75**, 3977–3980 (1995).

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[36]Equation (A4) can be written $H^2(\Omega-1) = k'/R^2$ where Ω is the ratio of actual to critical energy density (the critical energy density being that which makes space "flat," $k'=0$). Since departures of Ω from unity would grow rapidly with time, the difference between Ω and 1 in the very early universe must have been exceedingly small if not exactly zero. Hence we make negligible error if we set $k'=0$ in this calculation.