GABARITO TERCEIRA LISTA

PROBLEMA 1

$$\varepsilon_{\Lambda 0} = \varepsilon_{cr,0} = 5200 \frac{MeV}{m^3}$$

$$E = \varepsilon_{\Lambda 0} \times V = 5200 \frac{MeV}{m^3} \times \frac{4\pi R^3}{3} \quad , \text{ onde } R = 1UA$$

$1UA = 1,5 \times 10^{11}\, m$

$$\Rightarrow E = 5200 \times \frac{4\pi}{3} (1,5 \times 10^{11})^3\, MeV = 7,4 \times 10^{37}\, MeV$$

$1\, MeV = 1,6 \times 10^{-13}\, J \quad \Rightarrow \boxed{E = 11,8 \times 10^{24}\, J}$

Energia de repouso do sol: $E_\odot = M_\odot c^2$

$$\left. \begin{array}{l} M_\odot = 2 \times 10^{30}\, Kg \\ c = 3 \times 10^8\, m/s \end{array} \right\} \Rightarrow \boxed{E_\odot = 18 \times 10^{46}\, J}$$

Comparando esses dois resultados, não esperamos que a constante cosmológica tinha efeito significativo sobre o movimento dos planetas, mesmo que sua densidade de energia seja igual à densidade crítica. Como os dois resultados diferem de 22 ordens de grandeza, essa conclusão continua válida

PROBLEMA 2

Universo de Einstein: (estático)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\varepsilon_{ME} + 3P_M) + \frac{\Lambda}{3} = 0$$

ε_M : densidade de energia da matéria no Universo de Einstein.

P_M : pressão da matéria $= 0$

$$\Rightarrow \quad -\frac{4\pi G}{3c^2}\varepsilon_{ME} + \frac{\Lambda}{3} = 0$$

$$\Lambda = 4\pi G\, \varepsilon_{ME}/c^2$$

Se parte da matéria é convertida em radiação:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon_M + 3\underbrace{P_M}_{0} + \varepsilon_R + 3P_R) + \frac{\Lambda}{3}$$

ε_R: densidade de energia da radiação

P_R: pressão da radiação $= \varepsilon_R/3$

$$\Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^3}(\varepsilon_M + \varepsilon_R + \varepsilon_R - \varepsilon_{ME})$$

Para que a energia seja conservada, $\varepsilon_M + \varepsilon_R = \varepsilon_{ME}$.

$$\Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^3}\varepsilon_R < 0$$

O Universo começa a contrair!

PROBLEMA 3

Razão piezoenergética $\omega = \frac{P}{\varepsilon}$.

$$\varepsilon = nE = n\left[m^2c^4 + \frac{h^2c^2}{\lambda^2}\right]^{1/2}$$

$P = ?$

Usamos $dQ = dE + PdV$, com $dQ = 0$ em um Universo perfeitamente homogêneo.

Então, $P = -\frac{\partial E}{\partial V}$.

$$\lambda(t) = \lambda_0 \, a(t)$$

$$V(t) = \frac{4\pi}{3} r_0^3 a^3(t) \equiv V_0 \, a^3(t)$$

$$\Rightarrow \lambda = \lambda_0 \left(\frac{V}{V_0} \right)^{1/3}$$

Substituindo na expressão para E:

$$E = \left[m^2 c^4 + \frac{h^2 c^2}{\lambda_0^2} \left(\frac{V_0}{V} \right)^{2/3} \right]^{1/2}$$

$$P = -\frac{\partial E}{\partial V} = -\frac{1}{2E} \frac{h^2 c^2}{\lambda_0^2} V_0^{2/3} \left(-\frac{2}{3} \frac{1}{V^{2/3}} \frac{1}{V} \right)$$

$$= + \frac{h^2 c^2}{3E} \frac{1}{\lambda^2} \frac{1}{V}$$

Mas $\varepsilon = \frac{E}{V} \Rightarrow V = \frac{E}{\varepsilon}$: $P = \frac{h^2 c^2}{E^2 \lambda^2} \frac{\varepsilon}{3}$

$$\Rightarrow w = \frac{P}{\varepsilon} = \frac{1}{3} \frac{h^2 c^2}{E^2 \lambda^2}$$

No limite altamente relativístico: $E \to \frac{hc}{\lambda}$

$\Rightarrow w \to 1/3$.

No limite não-relativístico: $\lambda \to \infty$, $E \to mc^2$

$\Rightarrow w \to 0$.

PROBLEMA 4

a) $\Omega = \dfrac{\varepsilon}{\varepsilon_c}$ $\varepsilon_c \equiv \dfrac{3c^2}{8\pi G} H^2(t) \Rightarrow \varepsilon_c = \varepsilon_c(t)$

$$\frac{d\Omega}{dt} = \frac{d\varepsilon}{dt}\frac{1}{\varepsilon_c} - \frac{\varepsilon}{\varepsilon_c^2}\frac{d\varepsilon_c}{dt}$$

$$= \underbrace{\frac{d\varepsilon}{dt}}\frac{1}{\varepsilon_c} - \frac{\Omega}{\varepsilon_c}\frac{3c^2}{8\pi G}\underbrace{\frac{dH}{dt}}2H$$

$\hookrightarrow H = \dfrac{\dot{a}}{a} \Rightarrow \dfrac{dH}{dt} = \dfrac{\ddot{a}}{a} - \left(\dfrac{\dot{a}}{a}\right)^2$

Pela equação do fluido:

$$\frac{d\varepsilon}{dt} = -3\frac{\dot{a}}{a}(\varepsilon + P) = -3H\varepsilon(1+w)$$

$$\Rightarrow \frac{d\Omega}{dt} = -3H(1+w)\frac{\varepsilon}{\varepsilon_c} - \frac{\Omega}{H^2}\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right]2H$$

$$= -3H(1+w)\Omega - \frac{2\Omega}{H}\left[\frac{\ddot{a}}{a} - H^2\right]$$

Usando a equação da aceleração:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\varepsilon(1+3w) = -\frac{H^2}{\varepsilon_c}\frac{\varepsilon}{2}(1+3w)$$

temos:

$$\frac{d\Omega}{dt} = -3H(1+w)\Omega - \frac{2\Omega}{H}\left[-\frac{H^2\Omega}{2}(1+3w) - H^2\right]$$

$$= -3H\Omega - 3Hw\Omega + H\Omega^2(1+3w) + 2\Omega H$$

$$= -H\Omega - 3H\Omega w + H\Omega^2(1+3w)$$

$$= -H\Omega(1+3w) + H\Omega^2(1+3w)$$

$$\boxed{\frac{d\Omega}{dt} = H\Omega(\Omega-1)(1+3w)}$$

b) $H = \frac{\dot{a}}{a}$

$$\frac{dH}{dt} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{\ddot{a}}{a} - H^2$$

Usando a equação da aceleração:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\,\varepsilon(1+3w) = -\frac{1}{\varepsilon_c}\frac{H^2}{2}\,\varepsilon(1+3w),$$

temos:

$$\boxed{\frac{dH}{dt} = -\frac{\Omega H^2}{2}(1+3w) - H^2}$$

PROBLEMA 5

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\frac{\ddot{a}}{a}\frac{1}{H^2}$$

Usando a equação da aceleração:

$$\frac{\ddot{a}}{a} = -\frac{H^2\Omega}{2}(1+3w),$$

obtemos:

$$q = \frac{H^2\Omega}{2}(1+3w)\frac{1}{H^2}$$

$$\Rightarrow \boxed{q = \Omega(1+3w)/2}$$

PROBLEMA 6

a) $$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x-x_0) + \left.\frac{d^2f}{dx^2}\right|_{x=x_0} \frac{(x-x_0)^2}{2!} + \left.\frac{d^3f}{dx^3}\right|_{x=x_0} \frac{(x-x_0)^3}{3!} + \left.\frac{d^4f}{dx^4}\right|_{x=x_0} \frac{(x-x_0)^4}{4!} + \dots$$

Expandindo $a(t)$ em torno de t_0, temos:

$$a(t) = a(t_0) + \left.\frac{da}{dt}\right|_{t_0} (t-t_0) + \left.\frac{d^2a}{dt^2}\right|_{t_0} \frac{(t-t_0)^2}{2!} + \left.\frac{d^3a}{dt^3}\right|_{t_0} \frac{(t-t_0)^3}{3!} + \left.\frac{d^4a}{dt^4}\right|_{t_0} \frac{(t-t_0)}{4!} + \dots$$

Substituimos: $a(t_0) = a_0 \equiv 1$ e $\left.\frac{da}{dt}\right|_{t_0} = \left.\frac{1}{a_0}\frac{da}{dt}\right|_{t_0} = H_0$

$$\Rightarrow a(t) = 1 + H_0(t-t_0) + \left.\frac{d^2a}{dt^2}\right|_{t_0} \frac{(t-t_0)^2}{2!} + \left.\frac{d^3a}{dt^3}\right|_{t_0} \frac{(t-t_0)^3}{3!} + \left.\frac{d^4a}{dt^4}\right|_{t_0} \frac{(t-t_0)^4}{4!}$$

Parâmetro de desaceleração:

$$q(t) \equiv -\frac{1}{a}\frac{d^2a}{dt^2}\left[\frac{1}{a}\frac{da}{dt}\right]^{-2} = -\frac{1}{a}\frac{d^2a}{dt^2} H(t)^{-2}$$

$$\Rightarrow q(t_0) = q_0 = -\left.\frac{d^2a}{dt^2}\right|_{t_0} H_0^{-2} \quad e \quad \left.\frac{d^2a}{dt^2}\right|_{t_0} = -q_0 H_0^2$$

Jerk:

$$j(t) \equiv \frac{1}{a}\frac{d^3a}{dt^3}\left[\frac{1}{a}\frac{da}{dt}\right]^{-3} = \frac{1}{a}\frac{d^3a}{dt} H(t)^{-3}$$

$$j(t_0) = j_0 = \left.\frac{d^3a}{dt^3}\right|_{t_0} H_0^{-3} \Rightarrow \left.\frac{d^3a}{dt^3}\right|_{t_0} = j_0 H_0^3$$

Snap:

$$s(t) \equiv \frac{1}{a}\frac{d^4a}{dt^4}\left[\frac{1}{a}\frac{da}{dt}\right]^{-4} = \frac{1}{a}\frac{d^4a}{dt^4}H(t)^{-4}$$

$$s(t_0) = s_0 = \left.\frac{d^4a}{dt^4}\right|_{t_0} H_0^{-4} \Rightarrow \left.\frac{d^4a}{dt^4}\right|_{t_0} = s_0 H_0^4$$

$$\Rightarrow a(t) \simeq 1 + H_0(t-t_0) - q_0H_0^2\frac{(t-t_0)^2}{2!} + j_0H_0^3\frac{(t-t_0)^3}{3!} + s_0H_0^4\frac{(t-t_0)^4}{4!}$$

b) $$z(t) = z(t_0) + \left.\frac{dz}{dt}\right|_{t_0}(t-t_0) + \left.\frac{d^2z}{dt^2}\right|_{t_0}\frac{(t-t_0)^2}{2!} +$$

$$+ \frac{d^3z}{dt^3}\frac{(t-t_0)^3}{3!} + \frac{d^4z}{dt^4}\frac{(t-t_0)^4}{4!} + \dots$$

$$z(t_0) \equiv z_0 = 0$$

$$1+z = \frac{1}{a} \Rightarrow z = \frac{1}{a} - 1 \quad e \quad \frac{dz}{dt} = \frac{d(1/a)}{dt}$$

$$\Rightarrow z(t) = \left.\frac{d(1/a)}{dt}\right|_{t_0}(t-t_0) + \left.\frac{d^2(1/a)}{dt^2}\right|_{t_0}\frac{(t-t_0)^2}{2!} + \left.\frac{d^3(1/a)}{dt^3}\right|_{t_0}\frac{(t-t_0)^3}{3!} +$$

$$+ \left.\frac{d^4(1/a)}{dt^4}\right|_{t_0}\frac{(t-t_0)^4}{4!} + \dots$$

$$\frac{d(1/a)}{dt} = -\frac{1}{a^2}\frac{da}{dt} = -\frac{1}{a}H(t) \Rightarrow \boxed{\left.\frac{d(1/a)}{dt}\right|_{t_0} = -H_0}$$

$$\frac{d^2(1/a)}{dt^2} = \frac{d}{dt}\frac{d(1/a)}{dt} = \frac{d}{dt}\left(-\frac{1}{a^2}\frac{da}{dt}\right) = +\frac{2}{a^3}\left(\frac{da}{dt}\right)^2 - \frac{1}{a^2}\frac{d^2a}{dt^2}$$

$$\Rightarrow \left.\frac{d^2(1/a)}{dt^2}\right|_{t_0} = 2H_0^2 - \left.\frac{d^2a}{dt^2}\right|_{t_0} = 2H_0^2 + q_0H_0^2$$

$$\boxed{\left.\frac{d^2(1/a)}{dt^2}\right|_{t_0} = H_0^2(2+q_0)}$$

$$\frac{d^3(1/a)}{dt^3} = \frac{d}{dt}\frac{d^2(1/a)}{dt^2} = \frac{d}{dt}\left[\frac{2}{a^3}\left(\frac{da}{dt}\right)^2 - \frac{1}{a^2}\frac{d^2a}{dt^2}\right]$$

$$= -\frac{6}{a^4}\left(\frac{da}{dt}\right)^3 + \frac{2}{a^3}\cdot 2\,\frac{da}{dt}\frac{d^2a}{dt^2} + \frac{2}{a^3}\frac{da}{dt}\frac{d^2a}{dt^2} - \frac{1}{a^2}\frac{d^3a}{dt^3}$$

$$= -\frac{6}{a^4}\left(\frac{da}{dt}\right)^3 + \frac{6}{a^3}\frac{da}{dt}\frac{d^2a}{dt^2} - \frac{1}{a^2}\frac{d^3a}{dt^3}$$

$$\Rightarrow \left.\frac{d^3(1/a)}{dt^3}\right|_{t_0} = -6H_0^3 - 6H_0^3 q_0 - j_0 H_0^3$$

$$\boxed{\left.\frac{d^3(1/a)}{dt^3}\right|_{t_0} = -H_0^3\left(6 + 6q_0 + j_0\right)}$$

$$\frac{d^4(1/a)}{dt^4} = \frac{d}{dt}\frac{d^3(1/a)}{dt^3} = \frac{d}{dt}\left[-\frac{6}{a^4}\left(\frac{da}{dt}\right)^3 + \frac{6}{a^3}\frac{da}{dt}\frac{d^2a}{dt^2} - \frac{1}{a^2}\frac{d^3a}{dt^3}\right]$$

$$= \frac{24}{a^5}\left(\frac{da}{dt}\right)^4 - \frac{18}{a^4}\left(\frac{da}{dt}\right)^2\frac{d^2a}{dt^2} - \frac{18}{a^4}\left(\frac{da}{dt}\right)^2\frac{d^2a}{dt^2} + \frac{6}{a^3}\left(\frac{d^2a}{dt^2}\right)^2 +$$

$$+ \frac{6}{a^3}\frac{da}{dt}\frac{d^3a}{dt^3} + \frac{2}{a^3}\frac{da}{dt}\frac{d^3a}{dt^3} - \frac{1}{a^2}\frac{d^4a}{dt^4}$$

$$= \frac{24}{a^5}\left(\frac{da}{dt}\right)^4 - \frac{36}{a^4}\left(\frac{da}{dt}\right)^2\frac{d^2a}{dt^2} + \frac{6}{a^3}\left(\frac{d^2a}{dt^2}\right)^2 + \frac{8}{a^3}\frac{da}{dt}\frac{d^3a}{dt^3} - \frac{1}{a^2}\frac{d^4a}{dt^4}$$

$$\left.\frac{d^4(1/a)}{dt^4}\right|_{t_0} = 24H_0^4 + 36H_0^2 q_0 H_0^2 + 6q_0^2 H_0^4 + 8H_0 j_0 H_0^3 - s_0 H_0^4$$

$$\boxed{\left.\frac{d^4(1/a)}{dt^4}\right|_{t_0} = H_0^4\left(24 + 36q_0 + 6q_0^2 + 8j_0 - s_0\right)}$$

$$\Rightarrow z(t) = H_0(t_0 - t) + H_0^2\frac{(2+q_0)}{2!}(t_0-t)^2 + H_0^3\frac{(6+6q_0+j_0)}{3!}(t_0-t)^3 +$$

$$+ H_0^4\frac{(24 + 36q_0 + 6q_0^2 + 8j_0 - s_0)}{4!}(t_0-t)^4 + \ldots$$

c) Para inverter a série do item b, podemos escrevê-la da seguinte maneira:

$$z(t) = c_1 \Delta t + c_2 \Delta t^2 + c_3 \Delta t^3 + c_4 \Delta t^4 + \ldots, \qquad (i)$$

onde $\Delta t = (t_0 - t)$ e

$$\begin{cases} c_1 = H_0 \\ c_2 = \dfrac{H_0^2}{2!}(2+q_0) \\ c_3 = \dfrac{H_0^3}{3!}(6+6q_0+j_0) \\ c_4 = \dfrac{H_0^4}{4!}(24+36q_0+6q_0^2+8j_0-s_0) \end{cases}$$

Queremos obter uma série da forma:

$$\Delta t = a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots \qquad (ii)$$

Podemos então substituir (ii) em (i), para obter os valores dos coeficientes a_i:

$$\Delta t^2 = (a_1 z + a_2 z^2)^2 + 2(a_1 z + a_2 z^2)(a_3 z^3 + a_4 z^4) + (a_3 z^3 + a_4 z^4)^2 + \ldots$$

$$= a_1^2 z^2 + 2a_1 a_2 z^3 + a_2^2 z^4 + 2a_1 a_3 z^4 + O(z^5)$$

$$\Delta t^3 = \Delta t^2 . \Delta t = (a_1^2 z^2 + 2a_1 a_2 z^3 + a_2^2 z^4 + 2a_1 a_3 z^4) \times (a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4)$$

$$= a_1^3 z^3 + a_1^2 a_2 z^4 + 2a_1^2 a_2 z^4 + O(z^5)$$

$$\Delta t^4 = \Delta t^3 . \Delta t = (a_1^3 z^3 + a_1^2 a_2 z^4 + 2a_1^2 a_2 z^4) \times (a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4)$$

$$= a_1^4 z^4 + O(z^5)$$

substituindo na (i):

$$z = c_1 a_1 z + c_1 a_2 z^2 + c_1 a_3 z^3 + c_1 a_4 z^4 + c_2 a_1^2 z^2 + 2 c_2 a_1 a_2 z^3 +$$
$$+ c_2 a_2^2 z^4 + c_2 2 a_1 a_3 z^4 + c_3 a_1^3 z^3 + c_3 a_1^2 a_2 z^4 + c_3 2 a_1^2 a_2 z^4 +$$
$$+ c_4 a_1^4 z^4 + O(z^5)$$

$$z = c_1 a_1 z + (c_1 a_2 + c_2 a_1^2) z^2 + (c_1 a_3 + 2 c_2 a_1 a_2 + c_3 a_1^3) z^3 +$$
$$+ (c_1 a_4 + c_2 a_2^2 + c_2 2 a_1 a_3 + 3 c_3 a_1^2 a_2 + c_4 a_1^4) z^4 + O(z^5)$$

$$\Rightarrow \quad c_1 a_1 = 1$$

$$\boxed{a_1 = 1/c_1} = 1/H_0$$

$$c_1 a_2 + c_2 a_1^2 = 0$$

$$c_1 a_2 = -\frac{c_2}{c_1^2}$$

$$\boxed{a_2 = -c_2/c_1^3} = -(2+q_0)/2H_0$$

$$c_1 a_3 + 2 c_2 a_1 a_2 + c_3 a_1^3 = 0$$

$$c_1 a_3 = -2 c_2 a_1 a_2 - c_3 a_1^3 = +2 \frac{c_2 c_2}{c_1^4} - \frac{c_3}{c_1^3}$$

$$\boxed{a_3 = \frac{2c_2^2}{c_1^5} - \frac{c_3}{c_1^4}} = \frac{(2+q_0)^2}{2H_0} - \frac{(6 + 6q_0 + j_0)}{6H_0}$$

$$= \frac{6 + 6q_0 + 3q_0^2 - j_0}{6H_0}$$

$$c_1 a_4 + c_2 a_2^2 + c_2 2 a_1 a_3 + 3 c_3 a_1^2 a_2 + c_4 a_1^4 = 0$$

$$c_1 a_4 = -c_2 a_2^2 - 2 c_2 a_1 a_3 - 3 c_3 a_1^2 a_2 - c_4 a_1^4$$

$$= -\frac{c_2^3}{c_1^6} - \frac{2c_2}{c_1}\left(\frac{2c_2^2}{c_1^5} - \frac{c_3}{c_1^4}\right) + 3 \frac{c_3}{c_1^2} \frac{c_2}{c_1^3} - \frac{c_4}{c_1^4}$$

$$a_4 = -\frac{c_2^3}{c_1^7} - 4\frac{c_2^3}{c_1^7} + 2\frac{c_3 c_2}{c_1^6} + 3\frac{c_3 c_2}{c_1^6} - \frac{c_4}{c_1^5}$$

$$\boxed{a_4 = -\frac{5c_2^3}{c_1^7} + 5\frac{c_3 c_2}{c_1^6} - \frac{c_4}{c_1^5}}$$

$$a_4 = -\frac{5}{H_0^7}\frac{H_0^6}{8}(2+q_0)^3 + \frac{5}{H_0^6}\frac{H_0^5}{12}(2+q_0)(6+6q_0+j_0)$$

$$-\frac{H_0^4}{24}\frac{1}{H_0^5}(24+36q_0+6q_0^2+8j_0-s_0)$$

$$= -\frac{5}{8H_0}(4+4q_0+q_0^2)(2+q_0) + \frac{5}{12H_0}(12+12q_0+2j_0+$$

$$+6q_0+6q_0^2+q_0j_0) - \frac{1}{24H_0}(24+36q_0+6q_0^2+8j_0-s_0)$$

$$= -\frac{15}{24H_0}(\cancel{8}+8q_0+2q_0^2+4q_0+4q_0^2+q_0^3) + \frac{10}{24H_0}(\cancel{12}+18q_0+$$

$$+6q_0^2+2j_0+q_0j_0) - \frac{1}{24H_0}(24+36q_0+6q_0^2+8j_0-s_0)$$

$$= -\frac{15}{24H_0}(12q_0+6q_0^2+q_0^3) + \frac{10}{24H_0}(18q_0+6q_0^2+2j_0+q_0j_0)$$

$$-\frac{1}{24H_0}(24+36q_0+6q_0^2+8j_0-s_0)$$

$$= -\frac{1}{24H_0}(+24+36q_0+36q_0^2+15q_0^3-10q_0j_0-12j_0-s_0)$$

$$\Rightarrow H_0(t_0-t) = z(t) - \frac{(2+q_0)}{2}z(t)^2 + \frac{(6+6q_0+3q_0^2-j_0)}{6}z(t)^3 +$$

$$-\frac{1}{24}(24+36q_0+36q_0^2+15q_0^3-10q_0j_0-12j_0-s_0)z(t)^4+\dots$$