GABARITO

5ª LISTA DE PROBLEMAS

PROBLEMA 1

Caso geral:

$$H_0 t_0 = \int_0^{a_0=1} \frac{da}{\left[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0}\, a^2 + (1-\Omega_0)\right]^{1/2}}$$

Para $\Omega_{r,0} = 0$, $\Omega_{\Lambda,0} = 0$ e $\Omega_{m,0} = \Omega_0$:

$$H_0 t_0 = \int_0^1 \frac{da}{\left[\Omega_0/a + (1-\Omega_0)\right]^{1/2}} \qquad \left(\begin{array}{l}\Omega_0 < 1 \\ \Rightarrow 1-\Omega_0 > 0\end{array}\right)$$

$$= \frac{1}{\sqrt{\Omega_0}} \int_0^1 \frac{\sqrt{a}\, da}{\left[1 + \left(\frac{1-\Omega_0}{\Omega_0}\right) a\right]^{1/2}}$$

Chamando $C = \dfrac{1-\Omega_0}{\Omega_0}$ e $A^2 = Ca$,

Temos $2A\,dA = C\,da$ e $\sqrt{a} = \dfrac{A}{\sqrt{C}}$

$$\Rightarrow H_0 t_0 = \frac{1}{\sqrt{\Omega_0}} \int_0^{\sqrt{C}} \frac{A}{\sqrt{C}} \frac{2A}{C} \frac{dA}{\left[1 + A^2\right]^{1/2}}$$

$$= \frac{2}{C^{3/2}} \frac{1}{\sqrt{\Omega_0}} \int_0^{\sqrt{C}} \frac{A^2\, dA}{\left[1 + A^2\right]^{1/2}}$$

Chamando $A = \operatorname{senh}\theta \Rightarrow 1 + A^2 = 1 + \operatorname{senh}^2\theta = \cosh^2\theta$

$dA = \cosh\theta\, d\theta$

$$\Rightarrow H_0 t_0 = \frac{2}{C^{3/2}} \frac{1}{\sqrt{\Omega_0}} \int_{\operatorname{senh}^{-1} 0}^{\operatorname{senh}^{-1}\sqrt{C}} \frac{\operatorname{senh}^2\theta\, \cancel{\cosh\theta}\, d\theta}{\cancel{\cosh\theta}}$$

$$H_0 t_0 = \frac{2}{c^{3/2}} \frac{1}{\sqrt{\Omega_0}} \int_0^{\operatorname{senh}^{-1}\sqrt{c}} \frac{e^{2\theta} - 2 + e^{2\theta}}{4} \, d\theta$$

$$= \frac{1}{2c^{3/2}} \frac{1}{\sqrt{\Omega_0}} \left[\frac{e^{2\theta}}{2} - 2\theta - \frac{e^{-2\theta}}{2} \right]_0^{\operatorname{senh}^{-1}\sqrt{c}}$$

$$= \frac{1}{2c^{3/2}} \frac{1}{\sqrt{\Omega_0}} \Big[\underbrace{\operatorname{senh} 2\theta}_{\substack{\| \\ 2 \operatorname{senh}\theta \cosh\theta}} - 2\theta \Big]_0^{\operatorname{senh}^{-1}\sqrt{c}}$$

$$= \frac{1}{c^{3/2}} \frac{1}{\sqrt{\Omega_0}} \Big[\underbrace{\operatorname{senh}\theta \cosh\theta}_{\substack{\| \\ \sqrt{c}\sqrt{1+c}}} - \theta \Big]_0^{\operatorname{senh}^{-1}\sqrt{c}} \qquad \theta \to \operatorname{senh}^{-1}\sqrt{c}$$

$$\Rightarrow H_0 t_0 = \frac{1}{c^{3/2}} \frac{1}{\sqrt{\Omega_0}} \left[\sqrt{c}\sqrt{1+c} - \operatorname{senh}^{-1}\sqrt{c} \right]$$

Para escrever em termos do $\cosh^{-1}$, como a Ryden:

$$\begin{cases} \operatorname{senh}^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right) \\ \cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right) \to x \geq 1 \end{cases}$$

No nosso caso $x = \sqrt{c} = \left(\frac{1-\Omega_0}{\Omega_0} \right)^{1/2} = \Big(\underbrace{\frac{1}{\Omega_0}}_{>1} - 1 \Big)^{1/2}$ >1, porque $\Omega_0 < 1$

$$\Rightarrow \operatorname{senh}^{-1}\sqrt{c} = \ln\left[\sqrt{c} + \sqrt{c+1} \right]$$

$$= \ln\left[\sqrt{c} \left(1 + \sqrt{c+1}/\sqrt{c} \right) \right]$$

$$= \ln\sqrt{c} + \ln\left[1 + \sqrt{\frac{c+1}{c}} \right]$$

$$= \frac{1}{2}\ln c + \frac{1}{2}\ln\left[1 + \sqrt{\frac{c+1}{c}} \right]^2$$

$$= \frac{1}{2}\ln c + \frac{1}{2}\ln\left[1 + 2\sqrt{\frac{c+1}{c}} + \frac{c+1}{c} \right]$$

$$= \frac{1}{2} \ln \left[c \left(1 + 2\sqrt{\frac{c+1}{c}} + \frac{c+1}{c} \right) \right]$$

$$= \frac{1}{2} \ln \left[2c + 1 + \sqrt{\frac{4c+4}{c}}\, c \right] = \frac{1}{2} \ln \left[(2c+1) + \sqrt{4c^2+4c} \right]$$

$$= \frac{1}{2} \ln \left[(2c+1) + \sqrt{(2c+1)^2 - 1} \right]$$

$$= \frac{1}{2} \cosh^{-1} (2c+1)$$

$$\Rightarrow H_0 t_0 = \frac{1}{c^{3/2}} \frac{1}{\sqrt{\Omega_0}} \left[\sqrt{c}\sqrt{1+c} - \frac{1}{2} \cosh^{-1} (2c+1) \right]$$

substituindo $c = \frac{1-\Omega_0}{\Omega_0}$

$$H_0 t_0 = \frac{\Omega_0}{1-\Omega_0} \frac{1}{\sqrt{\Omega_0}} \sqrt{\frac{1}{\Omega_0}} - \frac{\Omega_0^{3/2}}{(1-\Omega_0)^{3/2}} \frac{1}{\sqrt{\Omega_0}} \frac{1}{2} \cosh^{-1} \left(\frac{2-\Omega_0}{\Omega_0} \right)$$

$$\Rightarrow \boxed{H_0 t_0 = \frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \cosh^{-1} \left(\frac{2-\Omega_0}{\Omega_0} \right)}$$

PROBLEMA 2

Uma componente com razão piezoenergética w obedece:

$$\varepsilon(a) = \varepsilon_0\, a^{-3(1+w)}$$

Para matéria não relativística:

$$w = 0$$

$$\Rightarrow \varepsilon_m(a) = \varepsilon_{m,0}\, a^{-3}$$

Para o campo de quintessência:

$$w = -\frac{1}{2} \Rightarrow \varepsilon_Q(a) = \varepsilon_{Q,0}\, a^{-3/2}$$

$\Rightarrow$ Quando $\varepsilon_m(a) = \varepsilon_Q(a)$ temos:

$$\varepsilon_{m,0}\, a_{mQ}^{-3} = \varepsilon_{Q,0}\, a_{mQ}^{-3/2}$$

$$\boxed{a_{mQ} = \left(\frac{\varepsilon_{m,0}}{\varepsilon_{Q,0}}\right)^{2/3}}$$

Equação de Friedmann;

$$H_0 t = \int_0^a \left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}\, a^2 + (1-\Omega_0) + \Omega_{Q,0}\sqrt{a}\right]^{-1/2} da$$

Para um Universo que contém somente matéria e quintessência:

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{m,0}/a + \Omega_{Q,0}\sqrt{a}}}$$

$$= \int_0^a \frac{\sqrt{a}\; da}{\sqrt{\Omega_{m,0} + \Omega_{Q,0}\, a^{3/2}}}$$

$$= 2\,\frac{2}{3}\,\frac{1}{\Omega_{Q,0}} \sqrt{\Omega_{m,0} + \Omega_{Q,0}\, a^{3/2}}\,\Bigg|_0^a$$

$$= \frac{4}{3}\,\frac{1}{\Omega_{Q,0}} \sqrt{\Omega_{m,0} + \Omega_{Q,0}\, a^{3/2}} - \frac{4}{3}\,\frac{\sqrt{\Omega_{m,0}}}{\Omega_{Q,0}}$$

Para obter $a(t)$:

$$\frac{4}{3}\,\frac{1}{\Omega_{Q,0}} \sqrt{\Omega_{m,0} + \Omega_{Q,0}\, a^{3/2}} = H_0 t + \frac{4}{3}\,\frac{\sqrt{\Omega_{m,0}}}{\Omega_{Q,0}}$$

$$\sqrt{\Omega_{m,0} + \Omega_{Q,0}\, a^{3/2}} = H_0 t\,\frac{3}{4}\,\Omega_{Q,0} + \sqrt{\Omega_{m,0}}$$

$$\cancel{\Omega_{m,0}} + \Omega_{Q,0}\, a^{3/2} = H_0^2 t^2 \frac{9}{16}\,\Omega_{Q,0}^2 + \frac{3}{2} H_0 t\, \Omega_{Q,0}\sqrt{\Omega_{m,0}} + \cancel{\Omega_{m,0}}$$

$$a^{3/2} = H_0^2 t^2 \frac{9}{16} \Omega_{a,0} + \frac{3}{2} H_0 t \sqrt{\Omega_{m,0}}$$

$$\Rightarrow \boxed{a(t) = \left[H_0^2 t^2 \frac{9}{16} \Omega_{a,0} + \frac{3}{2} H_0 t \sqrt{\Omega_{m,0}} \right]^{2/3}}$$

Escrevendo em termos de a_{ma}:

$$a_{ma} = \left(\frac{\varepsilon_{m,0}}{\varepsilon_{a,0}}\right)^{2/3} = \left(\frac{\Omega_{m,0}}{\Omega_{a,0}}\right)^{2/3} \Rightarrow a_{ma}^{3/2} = \frac{\Omega_{m,0}}{\Omega_{a,0}}$$

$$\Rightarrow a(t) = \left[H_0^2 t^2 \frac{9}{16} \frac{\Omega_{m,0}}{a_{ma}^{3/2}} + \frac{3}{2} H_0 t \sqrt{\Omega_{m,0}} \right]^{2/3}$$

Se $a \ll a_{ma} \Rightarrow t$ será muito pequeno e o segundo termo de $a(t)$ será dominante:

$$a(t) \approx \left(\frac{3}{2} H_0 t \sqrt{\Omega_{m,0}}\right)^{2/3}$$

$\Rightarrow a(t) \propto t^{2/3}$, como é esperado numa fase dominada por matéria não-relativística.

Se $a \gg a_{ma}$, o primeiro termo será o dominante:

$$a(t) \approx \left(H_0 t \frac{3}{4} \frac{\sqrt{\Omega_{m,0}}}{a_{ma}^{3/4}}\right)^{4/3}.$$

PROBLEMA 3

$$H_0 t_0 = \int_0^1 \frac{da}{\left[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_\Lambda a^2 + (1-\Omega_0)\right]^{1/2}}$$

→ Para $\Omega_{r,0} = \Omega_{\Lambda,0} = 0$ e $\Omega_{m,0} = \Omega_0 = 1$, o programa obtem:

$$\boxed{t_0 = \frac{2}{3}\frac{1}{H_0}},$$

que está de acordo com a expressão (5.48) da Ryden:

$$t_0 = \frac{2}{3(1+w)} H_0^{-1},$$

se substituirmos $w=0$.

→ Para $\Omega_{\Lambda,0} = \Omega_{m,0} = 0$ e $\Omega_{r,0} = \Omega_0 = 1$, temos:

$$\boxed{t_0 = \frac{1}{2}\frac{1}{H_0}},$$

que novamente está de acordo com a expressão (5.48) para $w = 1/3$.

→ Para o modelo padrão:

$$\boxed{t_0 \simeq 0{,}96 \frac{1}{H_0}}$$

PROBLEMA 4

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_K(r)^2 d\Omega^2 \right]$$

- Para um fóton: $c^2 dt^2 = a^2 dr^2$

$$\Rightarrow \quad dr = \pm \frac{c}{a} dt$$

Usando $H = \frac{da}{dt}\frac{1}{a}$, obtemos $dr = \pm \frac{c}{a} dt \, \frac{da}{dt} \frac{1}{Ha}$

$$\Rightarrow \quad dr = \pm \frac{c}{H} \frac{da}{a^2}$$

Como $1+z = 1/a$, temos $dz = -da/a^2$

$$\Rightarrow \quad dr = \pm \frac{c}{H} dz$$

Escolhendo o sinal positivo; $dr = \frac{c}{H} dz = \underbrace{\frac{c}{H_0}}_{D_{H,0}} \underbrace{\frac{H_0}{H}}_{1/E(z)} dz$