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Another look at Helmholtz's model for the gravitational contraction of the Sun

A C Tort¹ and F Nogarol²

¹ Instituto de Física Universidade Federal do Rio de Janeiro Caixa Postal 68.528,
CEP 21941-972 Rio de Janeiro, Brazil

² Centro Federal de Educação Tecnológica, Estrada de Adrianópolis, 1317 Santa Rita,
CEP 26041-271 Nova Iguaçu, Brazil

E-mail: tort@ufrj.br and fnogarol@gmail.com

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Abstract

We take another look at the Helmholtz model for the gravitational contraction of the Sun. We show that there are two other pedagogically useful ways of rederiving Helmholtz's main results that make use of Gauss's law, the concept of gravitational field energy and the work–kinetic energy theorem. An account of the energy balance involved in the gravitational contraction process is also discussed.

1. Introduction

In the 19th century, there was a great controversy on the question of how old the Earth was [1–3]. See also [4] for a pedagogically motivated introduction to this subject. Soon it was realized that an important related issue, perhaps the most important of them all, was the determination of the age of the Sun [5]. This is so because it was plausible (and still is) to suppose that the Earth could not be older than the Sun. In 1856, the renowned German physicist Hermann von Helmholtz (1821–1894) proposed a simple model for the formation of the Sun whose main ingredient was the gravitational contraction of gaseous matter due to the gravitational attraction of an initial homogeneous spherical protomass [7]. How the protomass was formed in the first place was not argued but we can well imagine instabilities of an initially very dispersed cloud of gas. The final outcome of this process, in the Helmholtz model, is a sphere of uniformly distributed gas. From a more realistic point of view, this is not possible, unless in a given moment for some critical value of the density of the collapsing gas there is a phase transition and the gas becomes an incompressible fluid. The real Sun is an extremely hot plasma that can be described reasonably well by the perfect gas law. This plasma is in hydrostatic equilibrium, i.e. in every point in its interior the gas pressure counterbalances the gravitational attraction. The density of the plasma is not uniform and depends radially on the distance to its centre varying from practically zero at the surface to approximately

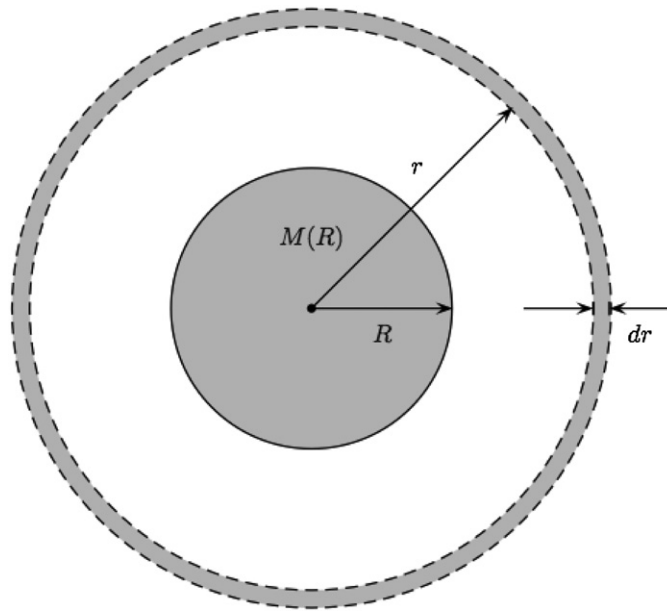


Figure 1. Helmholtz's model for the gravitational contraction of the Sun.

$160\,000\text{ kg m}^{-3}$ at the centre [6]. In any case, from his simple model, Helmholtz was able to estimate the age of the Sun. The numerical result of the calculation turned out to be wrong, but nevertheless the idea of gravitational contraction did not. Today we know that gravitational contraction is a main ingredient in the initial stage of the formation of the stars, the pre-main sequence stage. Gravitational contraction ignites the thermonuclear processes that determine the life and death of all the stars of the universe. From a pedagogical point of view, Helmholtz's calculation remains an excellent exercise in Newtonian gravitation and as such it can be exploited as a non-trivial example and presented to our students at the intermediate mechanics level. As we will argue here we can add features to the original calculation that may help to enhance the student's comprehension of the gravitational interaction, Gauss's theorem and the work–kinetic energy theorem. Also, for completeness, an account of the energy balance involved in the gravitational contraction process is recalled.

2. Helmholtz's model of the Sun

Consider a gaseous spherical protomass of radius R and mass $M(R)$ uniformly distributed in a region of space whose volume is $(4/3)\pi^3 R^3$. Suppose also that a spherical shell of mass dM is added to the mass $M(R)$. Let r be the radial distance between the shell to be added and the centre of the spherical distribution (see figure 1). In order to cancel out the attractive force that the protomass exerts on the shell, an external agent must supply an opposite force given by

$$dF(r) = \frac{GM(R) dM}{r^2}, \quad (1)$$

where $r > R > 0$. The mechanical work performed by this force on the shell is

$$d^2W = dF(r) dr = \frac{GM(R) dM dr}{r^2}. \quad (2)$$

Note that $dr < 0$ and this takes into account the fact that the applied force and displacement are opposite. If the shell is made to contract from an initial radius κR , where $\kappa \geq 1$, to R , the work done by the external agent reads

$$\begin{aligned} dW(R) &= GM(R) dM \int_{\kappa R}^R \frac{dr}{r^2} \\ &= -\frac{GM(R) dM}{R} \left(\frac{\kappa - 1}{\kappa} \right). \end{aligned} \quad (3)$$

In order to build a complete gaseous sphere, we must calculate the total work

$$W = \int_0^{R_\odot} dW(R), \quad (4)$$

where R_\odot in the Helmholtz model must be taken as an estimated value for the radius of the Sun. Assuming that the relation

$$\frac{M(R)}{R^3} = \frac{M_\odot}{R_\odot^3} \quad (5)$$

holds, where M_\odot in the Helmholtz model must also be taken as an estimated mass of the Sun, we can easily express $M(R)$ and dM as explicit functions of R and write

$$dW = -3G \frac{M_\odot^2}{R_\odot^6} \left(\frac{\kappa - 1}{\kappa} \right) R^4 dR. \quad (6)$$

Upon performing the integration, we obtain

$$W = -\frac{3}{5} \frac{GM_\odot^2}{R_\odot} \left(\frac{\kappa - 1}{\kappa} \right), \quad (7)$$

where

$$M_\odot = \rho_\odot \frac{4\pi}{3} R_\odot^3. \quad (8)$$

Note that the radius and the mass can in principle be determined by geometrical means and Kepler's laws. If the mass shells are brought from very far, that is, $\kappa \gg 1$, then

$$W = -\frac{3}{5} \frac{GM_\odot^2}{R_\odot}, \quad (9)$$

which is the result obtained by Helmholtz. Making use of the approximate values $R_\odot \approx 7 \times 10^8$ m, $M_\odot \approx 2 \times 10^{30}$ kg, we obtain

$$W \approx -2.3 \times 10^{41} \text{ J}. \quad (10)$$

A rough estimate of the age of the Sun in Helmholtz's model can be obtained if we divide the absolute value of W by the present energy output of the Sun, $P_\odot \approx 3.6 \times 10^{26} \text{ J s}^{-1}$. The result is about 20 million of years.

3. The energy associated with the gravitational field

Equation (7) can also be obtained in an alternative way that allows us to interpret the work done by the external force as stored in the gravitational field. We start with Gauss's law for the gravitational field \mathbf{g} :

$$\oint_S \mathbf{g} \cdot \hat{\mathbf{n}} da = -4\pi GM(R), \quad (11)$$

where \mathcal{S} is the boundary of the region \mathcal{R} , $\hat{\mathbf{n}}$ is the exterior unit normal vector, da is the measure of an element of area in \mathcal{S} , G is the gravitational constant and $M(\mathcal{R})$ is the gravitational mass enclosed by \mathcal{S} . The negative algebraic sign is due to the fact that $\mathbf{g} \cdot \hat{\mathbf{n}} da < 0$ for all points on \mathcal{S} . When the mass distribution has spherical symmetry, the flux of the gravitational field is easily calculated:

$$\oint_{\mathcal{S}} \mathbf{g} \cdot \hat{\mathbf{n}} da = -g(r)4\pi r^2, \quad (12)$$

where r is the radial distance of a point on \mathcal{S} to the centre of the distribution. It follows from Gauss's law that the intensity of the gravitational field is given by

$$g(r) = \frac{GM(r)}{r^2}, \quad r > 0. \quad (13)$$

For a gravitational mass M uniformly distributed in the interior of a spherical region of radius R , it is easily shown that

$$M(r) = M \frac{r^3}{R^3}, \quad 0 < r \leq R. \quad (14)$$

It follows that

$$\mathbf{g}(r) = \begin{cases} -\frac{GMr}{R^3} \hat{\mathbf{e}}_r, & 0 \leq r \leq R, \\ -\frac{GM}{r^2} \hat{\mathbf{e}}_r, & r \geq R. \end{cases} \quad (15)$$

Note that the gravitational field is continuous at $r = R$; hence, the superficial mass density of the distribution is zero as we can easily prove by making use once again of Gauss's law. The gravitational energy is given by

$$U_g = -\frac{1}{8\pi G} \int_{\mathcal{R}} \mathbf{g}^2 d^3x. \quad (16)$$

Note that the field gravitational energy is negative. Making use of the gravitational field given by equation (15) and of its spherical symmetry, we can write

$$U_g = -\frac{1}{8\pi G} \frac{G^2 M^2}{R^6} \int_0^R r^2 4\pi r^2 dr - \frac{1}{8\pi G} G^2 M^2 \int_R^\infty \frac{1}{r^4} 4\pi r^2 dr. \quad (17)$$

Evaluating the integrals, we obtain

$$U_g = -\frac{3}{5} \frac{GM^2}{R}, \quad (18)$$

which we construe as the gravitational energy associated with a homogeneous sphere of mass M and radius R . Let us apply this result to the gravitational contraction in the Helmholtz model. Suppose that initially the radius of the Sun in the model is κ times its final radius. Then, the initial gravitational energy is

$$U_i = -\frac{3}{5} \frac{GM_\odot^2}{\kappa R_\odot}, \quad (19)$$

with $\kappa > 1$. The final gravitational energy will be

$$U_f = -\frac{3}{5} \frac{GM_\odot^2}{R_\odot}. \quad (20)$$

Therefore, the variation of the gravitational energy reads

$$\Delta U_g = U_f - U_i = -\frac{3}{5} \frac{GM_\odot^2}{R_\odot} \left(\frac{\kappa - 1}{\kappa} \right). \quad (21)$$

In particular, if $\kappa \gg 1$ we recover equation (9). It follows that we can identify the variation of the gravitational energy stored in the field ΔU_g with the work W performed by the external agent in the creation of a homogeneous sphere. Moreover, in classical field theory, equations (21) and (9) are said to represent self-energy or binding energy of a homogeneous sphere. The binding energy is the energy that an external agent must supply in order to remove this sphere, shell after shell, to infinity until nothing of it is left behind.

4. The work–energy theorem and Helmholtz' model

Another instructive way of obtaining equations (9) or (21) is to make use of the work–energy theorem. The kinetic energy of the shell of mass dM is given by

$$dK = \frac{1}{2} dM v_r^2(r), \quad (22)$$

where $v_r(r)$ is the radial speed of a shell at a distance r from the centre of the spherical protomass. The radial acceleration reads

$$a_r = \frac{dv_r}{dt} = \frac{dv_r}{dr} \frac{dr}{dt} = v_r \frac{dv_r}{dr}. \quad (23)$$

We can integrate formally this equation from κR to an arbitrary radial distance r and write

$$\frac{1}{2} v_r^2(r) - \frac{1}{2} v_r^2(\kappa R) = \int_{\kappa R}^r a_r(r') dr'. \quad (24)$$

It is reasonable to suppose $v(\kappa R) = 0$; then,

$$\frac{1}{2} v_r^2(r) = \int_{\kappa R}^r a_r(r') dr'. \quad (25)$$

In order to determine the radial acceleration $a_r(r)$, consider equation (1). Taking into account the relation given by equation (5), we can recast equation (1) into the form

$$dF(r) = -\frac{GM_\odot}{R_\odot^2} \frac{R^3}{R_\odot} \frac{dM}{r^2} = -g_\odot \frac{R^3}{R_\odot} \frac{dM}{r^2}. \quad (26)$$

It follows that the radial acceleration can be written as

$$a_r(r) = -g_\odot \frac{R^3}{R_\odot} \frac{1}{r^2}. \quad (27)$$

We can now integrate equation (25) and obtain the kinetic energy per unit mass. The result is

$$\frac{1}{2} v_r^2(r) = \frac{dK}{dM} = g_\odot \frac{R^3}{R_\odot r} \left(1 - \frac{r}{\kappa R} \right). \quad (28)$$

Note that the mass of an arbitrary shell is conserved, that is, $dM = \rho(r)4\pi r^2 dr = \rho_\odot(R)4\pi R^2 dR$. When $r = R$,

$$\frac{1}{2} v_r^2(R) = g_\odot \frac{R^2}{R_\odot} \left(1 - \frac{1}{\kappa} \right). \quad (29)$$

It follows that

$$dK = \frac{1}{2} dM v_r(R) = g_\odot \frac{R^2}{R_\odot} \left(\frac{\kappa - 1}{\kappa} \right) \rho_\odot(R)4\pi R^2 dR. \quad (30)$$

Recalling that

$$\rho(R) = \frac{M(R)}{(4/3)\pi R^3}, \quad (31)$$

making use of equation (5) and taking this relation into equation (30), we have

$$dK = 3g_{\odot} \frac{R^4}{R_{\odot}^4} \left(\frac{\kappa - 1}{\kappa} \right) M_{\odot} dR. \quad (32)$$

The variation of the total kinetic energy is

$$\Delta K = 3g_{\odot} \frac{M_{\odot}}{R_{\odot}^4} \left(\frac{\kappa - 1}{\kappa} \right) \int_0^{R_{\odot}} R^4 dR = \frac{3}{5} g_{\odot} M_{\odot} R_{\odot} \left(\frac{\kappa - 1}{\kappa} \right), \quad (33)$$

or, if we prefer,

$$\Delta K = \frac{3}{5} \frac{GM_{\odot}^2}{R_{\odot}} \left(\frac{\kappa - 1}{\kappa} \right). \quad (34)$$

By the work–energy theorem, equation (34) corresponds to the work done by the gravitational force due to the protomass on all shells. This work is minus the work performed by the external agent of section 2.

5. The balance of energy and the contraction time

If we consider the Sun as a thermodynamical system in a state of (approximate) equilibrium, we must take into account internal thermal energy that we denote by E_{thermal} . If for the sake of simplicity we neglect losses due to blackbody radiation, the solar energy content that we observe reads

$$E = E_{\text{thermal}} + W_{\text{b}}, \quad (35)$$

where $W_{\text{b}} = -\frac{3}{5} \frac{GM_{\odot}^2}{R_{\odot}}$ is the binding energy in Helmholtz's model. The virial theorem of classical mechanics [8] states that for a self-interacting system for which the internal elements interact through a force law that depends on the n th power of the relative distance between two arbitrary elements i and j , that is, $F_{ij} \sim r_{ij}^n$, the time averages of its internal kinetic energy and self-potential energy are related through

$$\langle K \rangle = \frac{n+1}{2} \langle U \rangle. \quad (36)$$

If we identify $\langle K \rangle$ with the internal thermal energy and the self-potential energy with the binding energy, and set $n = -2$, we can write

$$2E_{\text{thermal}} + W_{\text{b}} = 0. \quad (37)$$

It follows that the energy of the Sun is

$$E = \frac{1}{2} W_{\text{b}} = -1.2 \times 10^{41} \text{J}. \quad (38)$$

The minus sign means that our star is in a bound state. Nevertheless, if in the model we go back in time, we can assume that the gas cloud was motionless and disassembled. Therefore, the total energy of the gas cloud in the very beginning of the contraction process was zero and by conservation of energy it must remain zero after the formation of the protomass and the final Sun. Something is missing. What is missing is that part of the energy generated by the gravitational contraction that is radiated away. The balance of energy should read

$$E_{\text{thermal}} + W_{\text{b}} + Q_{\text{rad}} = 0. \quad (39)$$

The gravitational contraction process generated $Q_{\text{rad}} = +1.2 \times 10^{41}$ J that were given away as radiation. If we divide this quantity by the present output of energy of the Sun we obtain a duration, the Kelvin–Helmholtz contraction time associated with the Helmholtz model:

$$T_{\text{K-H}} = \frac{1.2 \times 10^{41} \text{J}}{3.6 \times 10^{26} \text{J/s}} \approx 1.1 \times 10^7 \text{s}, \quad (40)$$

that is, about 10 million years. Astrophysicists interpret the Kelvin–Helmholtz contraction time not as the age of a star, but as an indication of its travel time from a completely disassembled state to an assembled one as a star of the main sequence powered by nuclear fuel. In a more complex but still pre-main sequence model, i.e. that stage in the life of the stars where nuclear processes are still turned off or are not relevant [9], a fraction of the radiated energy would be used to transform the atoms in the cloud (mainly hydrogen and helium) into a hot plasma. For more details on pre-main sequence stars, see, for instance, [9, 10].

6. Final remarks

It must be kept in mind that even not taking into account the nuclear processes that we now know to occur in the interior of the stars and other aspects of their internal structure such as the radial dependence of the density, temperature and pressure, Helmholtz's model, though very simple, brings about a correct concept. Its main ingredient, gravitational contraction, proved to be a valid one. Astrophysicists now know that gravitational contraction provides the initial energy to trigger the nuclear furnace in the interior of the stars.

The original Helmholtz gravitational contraction model of the Sun though too simple still remains a good example of how Newtonian mechanics and universal gravitation work. As such it can be used to illuminate the important concepts of gravitational energy, Gauss's law as applied to gravitation, the work–energy theorem and conservation of energy as we have tried to show here. Most of our students come to life when we deal in class with topics such as the age of the solar system, or how it was created. Helmholtz's model, though dated, is an excellent doorway into those subjects.

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