

Galileo's kinematical paradox and the role of resistive forces

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Abstract. We discuss Galileo's kinematical "paradox" taking into account the effects of sliding friction and of resistive forces proportional to velocity. We show that sliding friction eliminates the paradox but still allows for very simple synchronous curves. Perhaps surprisingly, Galileo's paradox is preserved when the resistive force is proportional to velocity.

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1. Introduction

Given the vertical circle shown in figure 1, suppose we release a body initially at rest at point A , letting it fall freely along the vertical diameter AC . At the same time we release another body from point B , also starting from rest, and have it slide down without friction along chord BC . Then, the two bodies reach point C at the same instant. Any chord will do; there is no restriction on point B other than being on the circle. This effect was described by Galileo in a letter to his friend Guidobaldo del Monte [1] and also in *Two New Sciences* [2]. It is sometimes called Galileo's theorem [3] or Galileo's paradox [4], and we will use the latter form as it conveys better the surprise that many people show when they first hear about it (of course there is no paradox, it's just something very non-intuitive). As other "paradoxes" have been ascribed to Galileo (the infinity paradox, the hydrostatic paradox) we refer to the one of interest here as the kinematical paradox.

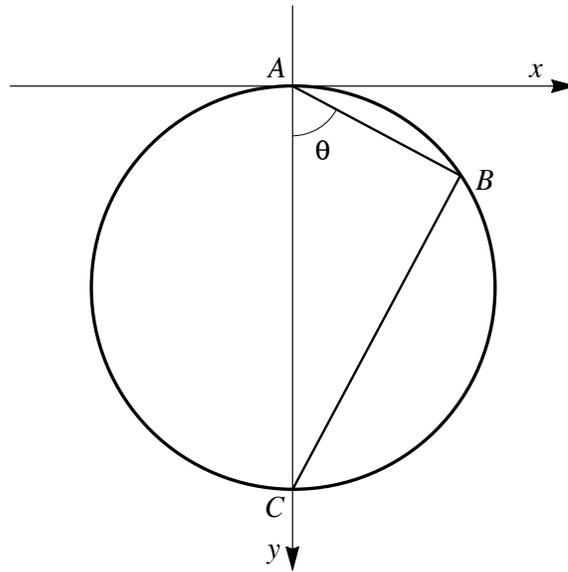


Figure 1. Galileo's paradox circle. Diameter AC is vertical and particles can move along chords like AB or BC . Angle θ is positive when point B is to the right of diameter AC .

Galileo's kinematical paradox has an interesting consequence: if several bodies initially at rest are released simultaneously from the top of the circle, each one sliding along a different chord starting at A (like AB in figure 1), then at a given instant their positions will be on an expanding circle whose centre moves down with acceleration $g/2$. This was already discussed by Galileo in [2], and a more pedagogically oriented presentation can be found in [5, 6].

It is natural to inquire how resistive forces affect Galileo's kinematical paradox. Here, we analyse the dynamical role of such forces in two cases: sliding friction and a resistive force proportional to velocity. We treat each case separately in sections 3 and 4, and study their combined effect in section 5. We also identify, in section 6, the class of resistive forces that preserve the paradox. Let us start by reviewing Galileo's

paradox.

2. Galileo's kinematical paradox: a brief review

Consider the vertical circle shown in figure 1 and a bead that slides without friction along chord AB , starting from rest at point A . At time t the position $s(t)$ of the bead along the chord, measured from point A , is given by

$$s = \frac{gt^2}{2} \cos \theta, \quad (1)$$

where the release time is $t = 0$ and θ is the angle $\angle CAB$ between the chord and the vertical direction. The time it takes for the bead to reach point B is, then,

$$t_{AB} = \sqrt{\frac{2s_{AB}}{g \cos \theta}}, \quad (2)$$

where s_{AB} is the length of chord AB . From Thales' theorem, $\angle ABC$ is a right angle, so that

$$s_{AB} = D \cos \theta \quad (3)$$

where D is the diameter of the circle. Hence,

$$t_{AB} = \sqrt{\frac{2D}{g}}, \quad (4)$$

independent of θ and, consequently, of the choice of point B . Beads going down along different chords will hit the rim of the circle at the same time. This is Galileo's kinematical paradox. A similar argument shows that the descent time along chord BC is the same for any choice of B , which was the original formulation of the paradox by Galileo in his letter to Guidobaldo del Monte [1].

Another way of presenting Galileo's kinematical paradox is, as we have already mentioned, through the expanding circle of simultaneity [5]. Consider the Cartesian axes shown in figure 1. The coordinates on these axes of a bead that slides along the chord AB are

$$x = s \sin \theta = \frac{gt^2}{4} \sin 2\theta, \quad (5)$$

and

$$y = s \cos \theta = \frac{gt^2}{4} (1 + \cos 2\theta), \quad (6)$$

where we have used the familiar trigonometric relations

$$2 \sin \theta \cos \theta = \sin 2\theta, \quad (7)$$

$$2 \cos^2 \theta = 1 + \cos 2\theta. \quad (8)$$

It is easily shown (see [5]) that equations (5) and (6) lead to

$$x^2 + \left(y - \frac{gt^2}{4}\right)^2 = \left(\frac{gt^2}{4}\right)^2, \quad (9)$$

which represents a circle with time-dependent radius given by

$$R(t) = \frac{gt^2}{4}. \quad (10)$$

The centre of this circle moves vertically along the diameter AC with acceleration $g/2$.

This means that, at a given time, particles starting from A along different chords all lie on a circle, the *synchronous curve* for this problem. In particular, at

$$\tau = \sqrt{\frac{2D}{g}}, \quad (11)$$

the time for free fall along the diameter AC , the synchronous circle coincides with the geometric circle ABC . Figure 2 shows the simultaneity circle for different values of t/τ . Links to video demonstrations of these effects can be found in [5, 6].

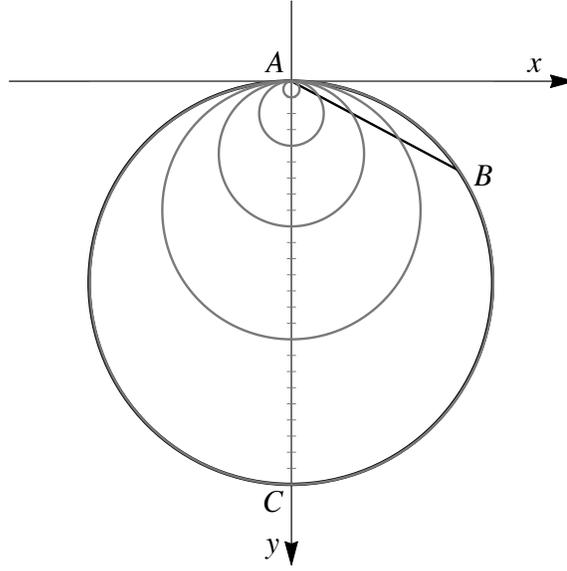


Figure 2. Synchronous curves for particles sliding down along chords originating at A with different orientations. Curves are shown for $t/\tau = 0.2, 0.4, 0.6, 0.8,$ and 1.0 , where τ is the time of free fall from A to C .

3. Sliding friction

Let us now consider the effect of resistive forces on Galileo's kinematical paradox and on the expanding synchronous circle. In a concrete realization of this system, the chords can be replaced, for instance, by thin wires and the circle by a bicycle wheel as discussed in [5]. Friction is then an unavoidable part of the dynamics and must be dealt with. We start by discussing the sliding friction on a bead that moves down on a chord. Suppose the kinematical friction coefficient between the bead and the chord (a plastic or metallic wire) is μ_k . For positive angles θ between the chord and the vertical direction, the equation of motion reads

$$\ddot{s} = g (\cos \theta - \mu_k \sin \theta). \quad (12)$$

Note that in order for the bead to slide down we must have $\theta \leq \theta_{\max}$, where

$$\theta_{\max} = \arctan(1/\mu_k). \quad (13)$$

For $\theta < 0$ we must change the sign of the second term in (12), otherwise the friction force will point in the wrong direction. In this case the equation of motion is

$$\ddot{s} = g(\cos\theta + \mu_k \sin\theta), \quad -\theta_{\max} < \theta < 0. \quad (14)$$

Assuming for the moment that $\theta \geq 0$, it is straightforward to solve (12) with initial conditions $s(0) = 0$ and $\dot{s}(0) = 0$, and find

$$s = \frac{gt^2}{2} (\cos\theta - \mu_k \sin\theta). \quad (15)$$

The time it takes for a bead to go down chord AB is, then,

$$t_{AB} = \sqrt{\frac{2s_{AB}}{g(\cos\theta - \mu_k \sin\theta)}} \quad (16)$$

and using (3) we obtain

$$t_{AB} = \sqrt{\frac{2D/g}{1 - \mu_k \tan\theta}}. \quad (17)$$

Similarly, for negative angles θ the result is

$$t_{AB} = \sqrt{\frac{2D/g}{1 + \mu_k \tan\theta}}. \quad (18)$$

We see that, because of sliding friction, beads that slide down the steeper chords (small $|\theta|$) reach the rim of the circle sooner than the ones following chords of milder slope (large $|\theta|$). Thus, Galileo's paradox no longer holds and the synchronous curve is not a circle anymore.

In spite of this, the synchronous curve still has a simple geometrical description: as we will see, it is the union of two circular arcs. To show this we write the Cartesian coordinates of the position of the bead as

$$x = s \sin\theta = \frac{gt^2}{4} [\sin 2\theta - \mu_k (1 - \cos 2\theta)], \quad (19)$$

and

$$y = s \cos\theta = \frac{gt^2}{4} [(1 + \cos 2\theta) - \mu_k \sin 2\theta], \quad (20)$$

where we have taken $\theta \geq 0$ and used (15) for $s(t)$. Eliminating θ from (19) and (20) we obtain

$$\left(x + \mu_k \frac{gt^2}{4}\right)^2 + \left(y - \frac{gt^2}{4}\right)^2 = \left(\frac{gt^2}{4}\right)^2 (1 + \mu_k^2), \quad (21)$$

the equation of a circle of radius

$$R(t) = \frac{gt^2}{4} \sqrt{1 + \mu_k^2}. \quad (22)$$

In contrast with the zero friction case, the centre of this circle is not on the y -axis. As t increases it moves down with acceleration $(g/2)\sqrt{1 + \mu_k^2}$ along a straight line that makes a negative angle $\theta = -\theta_c$ with the vertical direction, where

$$\theta_c = \arctan(\mu_k). \quad (23)$$

As we have considered only chords with inclination $\theta \geq 0$, the circle described by (21) gives the synchronous curve for $x \geq 0$ only. For negative values of θ a similar calculation provides the synchronous curve for $x < 0$, the result being

$$\left(x - \mu_k \frac{gt^2}{4}\right)^2 + \left(y - \frac{gt^2}{4}\right)^2 = \left(\frac{gt^2}{4}\right)^2 (1 + \mu_k^2). \quad (24)$$

Again this is an expanding circle, with the same radius as in the positive θ case. The centre of this circle also goes down following a straight line, with the same acceleration as before. The difference is that now this line makes a positive angle $\theta = \theta_c$ with the y -axis.

The complete synchronous curve is then the union of the $x > 0$ and $x < 0$ arcs of the circles defined in (21) and (24), respectively, resulting in a spindle-shaped contour. Figure 3 shows synchronous curves for $\mu_k = 0.5$. They are drawn for times $t/\tau = 0.2, 0.4, 0.6, 0.8$, and 1.0 , where τ is the time of free fall along the vertical diameter given in (11).

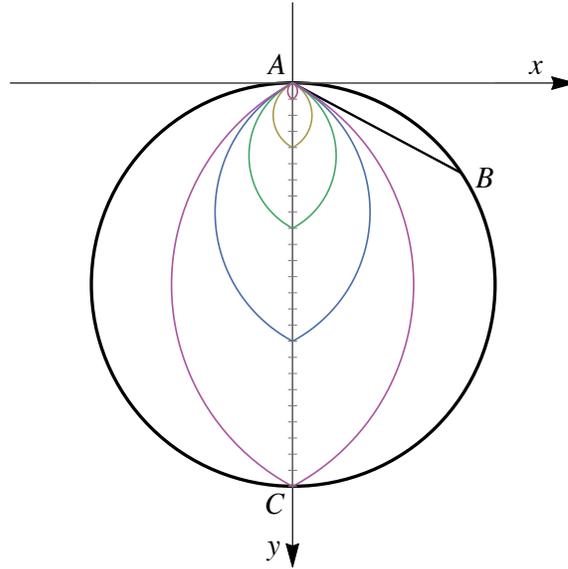


Figure 3. Synchronous curves for $\mu_k = 0.50$. The curves correspond to times $t/\tau = 0.2, 0.4, 0.6, 0.8$, and 1.0 , where τ is the time of free fall along the vertical diameter.

Another way of viewing the synchronous curves is attained by rescaling the x and y coordinates by the factor $gt^2/4$:

$$\tilde{x} = x/(gt^2/4), \quad \tilde{y} = y/(gt^2/4). \quad (25)$$

In the rescaled coordinates the circles (21) and (24) become time-independent and read, respectively,

$$(\tilde{x} + \mu_k)^2 + (\tilde{y} - 1)^2 = 1 + \mu_k^2, \quad (26)$$

and

$$(\tilde{x} - \mu_k)^2 + (\tilde{y} - 1)^2 = 1 + \mu_k^2. \quad (27)$$

These rescaled circles are shown by the dashed lines in figure 4. The arcs that form the time-independent synchronous curve in the rescaled coordinates are indicated by the solid lines. Both circles have radius $\sqrt{1 + \mu_k^2}$ and their centres have coordinates $(\tilde{x}, \tilde{y}) = (\pm \mu_k, 1)$. The circles cross on the vertical axis, at $\tilde{y} = 0$ and $\tilde{y} = 2$, the upper and lower limits of the synchronous curve. The scaling makes it easy to check how changes in the friction coefficient affect the synchronous curves. In particular we note that the simultaneity circle discussed in the previous section is recovered in the limit of zero friction.

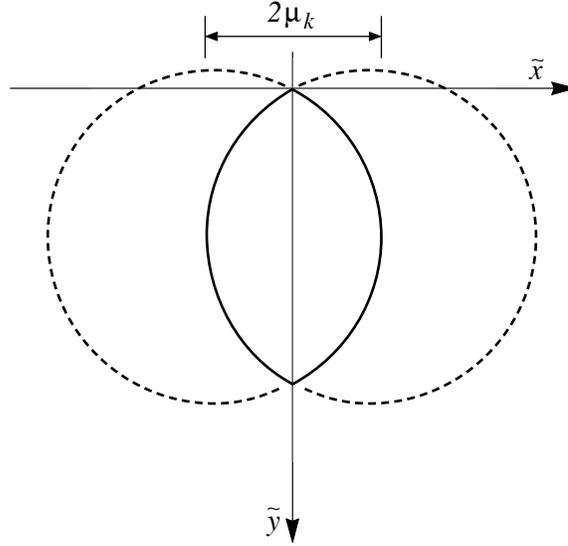


Figure 4. The time-independent synchronous curve in the rescaled coordinates (solid arcs). The dashed lines represent the circles defined by (26) and (27). The circles are displaced from each other by $2\mu_k$, have radius $\sqrt{1 + \mu_k^2}$ and cross at $\tilde{y} = 0$ and $\tilde{y} = 2$.

4. Motion under a resistive force proportional to velocity

Other types of dissipative forces can also produce simple synchronous curves and even restore Galileo's kinematical paradox. Let us consider the case of a resistive force linearly proportional to velocity. The equation of motion of a particle moving on chord AB (see figure 1) is, then,

$$\ddot{s} = g \cos \theta - \gamma \dot{s}, \quad (28)$$

where γ is the dissipation coefficient. With initial conditions $s(0) = 0$ and $\dot{s}(0) = 0$ we find that

$$s(t) = \left[\frac{g}{\gamma} t - \frac{g}{\gamma^2} (1 - e^{-\gamma t}) \right] \cos \theta. \quad (29)$$

Using (3) for the chord length s_{AB} , the descent time t_{AB} from A to B is obtained solving the equation

$$D = \frac{g}{\gamma} t_{AB} - \frac{g}{\gamma^2} (1 - e^{-\gamma t_{AB}}). \quad (30)$$

The angle θ does not appear in (30), implying that the descent time is the same for all chords. Thus, Galileo's kinematical paradox persists in the presence of a linear resistive force.

Because of Galileo's paradox, the synchronous curves are again circles, as in the case of no dissipation. The only difference is that, now, the radius of the circle of simultaneity increases more slowly due to the resistive force. Proceeding as before we obtain that the circle of simultaneity is given by

$$x^2 + [y - R(t)]^2 = R^2(t), \quad (31)$$

where

$$R(t) = \frac{g}{2\gamma}t - \frac{g}{2\gamma^2}(1 - e^{-\gamma t}). \quad (32)$$

In figure 5 we show the circles of simultaneity when the dissipation coefficient is such that $\gamma\tau = 0.50$. As before, τ is the time of free fall along the vertical diameter. The circles are drawn for $t/\tau = 0.2, 0.4, 0.6, 0.8,$ and 1.0 .

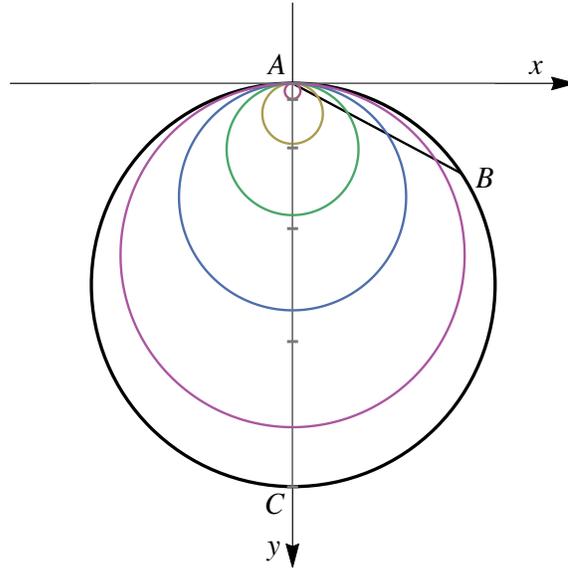


Figure 5. Synchronous circles for particles going down chords under the action of a linear resistive force. The dissipation coefficient is such that $\gamma\tau = 0.50$, where τ is the time of free fall along the vertical diameter. The curves correspond to times $t/\tau = 0.2, 0.4, 0.6, 0.8,$ and 1.0 . The marks on the y -axis indicate the position that a free falling body would have at these times.

It is interesting to note that the descent time t_{AB} can be written in terms of the Lambert W function [7, 8]. To show this we rewrite (30) as

$$F(\gamma t_{AB} - 1 - \gamma^2 D/g) = -e^{-1 - \gamma^2 D/g} \quad (33)$$

where

$$F(\omega) = \omega e^\omega. \quad (34)$$

Lambert's function $W(q)$ is the inverse of $F(\omega)$,

$$F(\omega) = q \quad \Rightarrow \quad \omega = W(q), \quad (35)$$

so that we can obtain t_{AB} from (33) as

$$t_{AB} = \frac{1}{\gamma} + \frac{\gamma D}{g} + \frac{1}{\gamma} W\left(-e^{-1-\gamma^2 D/g}\right). \quad (36)$$

A plot of $F(\omega)$ is shown in figure 6. The function has a minimum at $\omega = -1$, with $F(-1) = -1/e$. As ω increases, $F(\omega)$ decreases monotonically if $\omega \leq -1$ and increases monotonically if $\omega \geq -1$. These two parts of the curve are represented by dashed and solid lines in figure 6, each part defining a different branch of the inverse function $W(q)$. The *principal branch* $W_0(q)$ corresponds to $W(q) \geq -1$, and branch $W_{-1}(q)$ corresponds to $W(q) \leq -1$. (The complex Lambert function has an infinite number of branches [7], usually called W_n , $n = 0, \pm 1, \pm 2, \dots$) When there is no risk of confusion, W_0 is referred to as W .

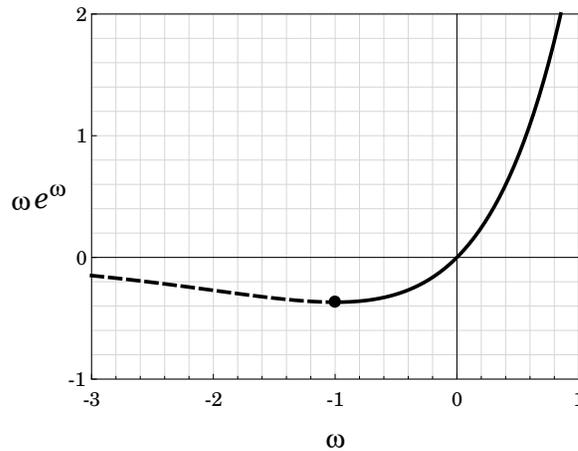


Figure 6. Plot of $F(\omega) = \omega e^\omega$. The monotonically decreasing and increasing branches are shown by the dashed and solid curves.

In order to compute t_{AB} from (36) we still have to determine which of the two real branches of W must be used. A simple way of doing this is by examining the small γ limit, which involves studying how $W(q)$ behaves in the vicinity of the branch point $q = -1/e$. Expanding $F(\omega)$ in a Taylor series about the minimum $\omega = -1$ we obtain, to second order, that

$$F(\omega) = -\frac{1}{e} + \frac{1}{2e}(\omega + 1)^2 + \dots \quad (37)$$

This truncated series is easily inverted and we find that for $q \approx -1/e$ the two branches of W are given by

$$W_0(q) \approx -1 + \sqrt{2(eq + 1)} \quad (38)$$

and

$$W_{-1}(q) \approx -1 - \sqrt{2(eq + 1)}. \quad (39)$$

Substituting these results in (36) we find that in the limit $\gamma \rightarrow 0$ the descent time is $t_{AB} = \pm \sqrt{2D/g}$, where the plus or minus sign corresponds to using W_0 or W_{-1} . Thus, as the descent time should always be positive, we must use the principal branch W_0 when computing t_{AB} from (36).

5. Motion under sliding friction and resistive force proportional to velocity

As a final case, let us consider the combined effects of sliding friction and a resistive force proportional to velocity. The equation of motion along chord AB is, then,

$$\ddot{s} = g(\cos \theta - \mu_k \sin \theta) - \gamma \dot{s} \quad (40)$$

where, as before, μ_k and γ are the kinetic friction and dissipation coefficients. We are assuming that $0 \leq \theta \leq \theta_{max}$, where θ_{max} is given by (13). For negative angles $-\theta_{max} \leq \theta < 0$ we must change the sign in front of μ_k , as discussed in section 3. For initial conditions $s(0) = 0$ and $\dot{s}(0) = 0$, the solution of (40) is

$$s(t) = \left[\frac{g}{\gamma} t - \frac{g}{\gamma^2} (1 - e^{-\gamma t}) \right] (\cos \theta - \mu_k \sin \theta). \quad (41)$$

Using (3) for the chord length s_{AB} , the descent time t_{AB} from A to B is given by the solution of

$$D = \left[\frac{g}{\gamma} t_{AB} - \frac{g}{\gamma^2} (1 - e^{-\gamma t_{AB}}) \right] (1 - \mu_k \tan \theta). \quad (42)$$

Again with help of Lambert's function we obtain

$$t_{AB} = \frac{1}{\gamma} W_0 \left[-\exp \left(-1 - \frac{D\gamma^2}{g(1 - \mu_k \tan |\theta|)} \right) \right] + \frac{1}{\gamma} + \frac{D\gamma}{g(1 - \mu_k \tan |\theta|)} \quad (43)$$

where the absolute value of θ takes care of the sign change for negative angles.

The descent time is θ -dependent, meaning that Galileo's paradox has disappeared and the synchronous curve is no longer a circle. Proceeding as in section 3 we find that the synchronous curve is given by

$$[x \pm X(t)]^2 + [y - Y(t)]^2 = R^2(t), \quad (44)$$

where the positive (negative) sign corresponds to positive (negative) values of θ , and

$$Y(t) = \frac{g}{2\gamma} t - \frac{g}{2\gamma^2} (1 - e^{-\gamma t}), \quad (45)$$

$$X(t) = \mu_k Y(t), \quad (46)$$

$$R(t) = \sqrt{1 + \mu_k^2} Y(t). \quad (47)$$

As in the case where only sliding friction is acting, the synchronous curve is spindle-shaped, being formed by two arcs. Figure 7 shows the synchronous curves for $\gamma\tau = 0.50$ and $\mu_k = 0.36$, drawn for times $t/\tau = 0.2, 0.4, 0.6, 0.8$, and 1.0 , where τ is the time of free fall along the vertical diameter.

Rescaling the x and y coordinates by $Y(t)$,

$$\tilde{x} = x/Y(t), \quad \tilde{y} = y/Y(t). \quad (48)$$

we obtain, once again, a very simple description of the synchronous curve:

$$(\tilde{x} \pm \mu_k)^2 + (\tilde{y} - 1)^2 = 1 + \mu_k^2. \quad (49)$$

This is the same curve we found when sliding friction was the only resistive force, see (26), (27) and figure 4.

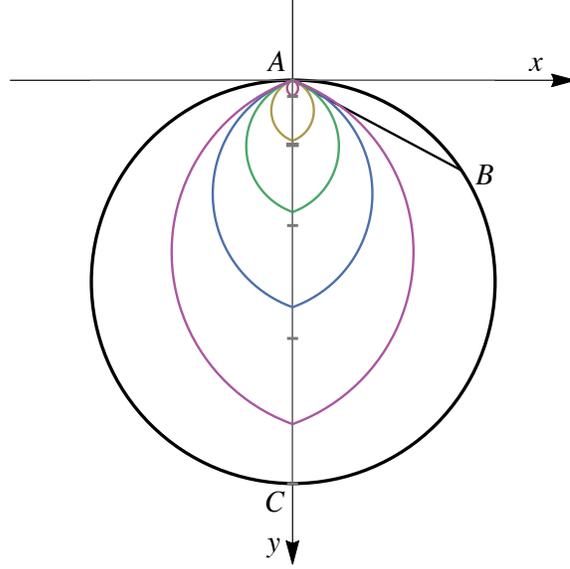


Figure 7. Synchronous curves for $\gamma\tau = 0.50$ and $\mu_k = 0.36$. The curves correspond to times $t/\tau = 0.2, 0.4, 0.6, 0.8,$ and 1.0 , where τ is the time of free fall along AC . The marks on the y -axis indicate the position that a free falling body would have at these times.

6. Which dissipative forces preserve Galileo's paradox?

Resistive forces may depend on the velocity in ways that are more complicated than the linear relation we have studied, and it is interesting to investigate if other functional forms will allow for Galileo's paradox. In order to do this, let us consider a resistive force given by some unspecified function $f(v)$ of the velocity v , so that the equation of motion of a body on a chord of inclination θ reads

$$\ddot{s} = g \cos \theta - f(\dot{s}). \quad (50)$$

As before, initial conditions are $s(0) = 0$ and $\dot{s}(0) = 0$. Galileo's paradox implies that the synchronous curves must be circles given by

$$x^2 + [y - R(t)]^2 = R^2(t), \quad (51)$$

where $2R(t)$ is the solution of (50) for $\theta = 0$. Substituting $x = s \sin \theta$ and $y = s \cos \theta$ for the coordinates in (51) we find that

$$s = 2R(t) \cos \theta. \quad (52)$$

Taking this result into the equation of motion (50) we obtain

$$\frac{f[2\dot{R}(t) \cos \theta]}{2 \cos \theta} = \frac{g}{2} - \ddot{R}(t). \quad (53)$$

The right-hand side of (53) is independent of θ , and so must be the left-hand side. It is easy to show that the latter happens only if $f(v) = \gamma v$, where γ is a constant. Thus, Galileo's paradox is preserved only in the case where the resistive force is proportional to the velocity.

7. Final remarks

We have studied the effects of dissipative forces on Galileo's kinematical paradox and the associated synchronous circle. More specifically, we investigated the role played by sliding friction and by a resistive force proportional to velocity. We found that sliding friction eliminates Galileo's paradox but still produces very simple synchronous curves, composed by circular arcs from two intersecting circles. In the case of a resistive force proportional to velocity, Galileo's paradox, somewhat surprisingly, is preserved and the synchronous curves keep the circular form. An interesting feature of this analysis is that it involves the application of the Lambert W function, a very useful multivalued function which perhaps is not as widely known as it should. We have also considered the combination of the two types of dissipative forces and found that, as in the case when sliding friction acts alone, Galileo's paradox breaks down but the synchronous curves have a simple geometrical description in terms of two circular arcs. These results suggest that Galileo's kinematical paradox persists only when the dissipative force depends linearly on the velocity, and we have shown that this is indeed the case.

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